FUNCTION[LEAST[x]]

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In[1]:= <<goedel54.11a; << tools.m

{Package Title: goedel54.11a 2004 February 11 at 5:30 p.m.

It is now: 2004 Feb 13 at 15:2

Loading Simplification Rules

TOOLS.M Revised 2004 January 3

weightlimit = 40

summary

If x is an antisymmetric relation, then the relations GREATEST[x] and LEAST[x] are functions. These facts had been proved 2000 November 24 using Otter, and added to the GOEDEL program at that time as conditional rules:

In[2]:= Begin["Goedel\'Private"];

This is Theorem GT–FU–2, proved 2000 November 24 using Otter:

In[3]:= InfoMatch[FUNCTION[GREATEST[x_]]]

Out[3]//TableForm=
  FUNCTION[GREATEST[x_]] := True ; subclass[intersection[x, inverse[x]], Id]

This is Theorem LT–FU, proved using Otter:

In[4]:= InfoMatch[FUNCTION[LEAST[x_]]]

Out[4]//TableForm=
  FUNCTION[LEAST[x_]] := True ; subclass[intersection[x, inverse[x]], Id]

The aim of this notebook is to rederive these two formulas using the GOEDEL program, and to supplement these conditional rewrite rules with clauses that can be used for reasoning. Such reasoning was used in the notebook WO–WRAP.NB dated 2004 January 24 in connection with a rederivation of Theorem WO–FU–LT which asserts that if x is a wellordering, then LEAST[x] is a function.

GREATEST

The rules for GREATEST[x] are the focus of this section. The starting point is the following derivation of Theorem GT–FU–1 that had also been proved 2000 November 24 using Otter:
SubstTest[implies, subclass[u, v], subclass[composite[u, w], composite[v, w]], {u -> GREATEST[x], v -> UB[x], w -> inverse[GREATEST[x]]}]


In[6]:= subclass[composite[GREATEST[x_], inverse[GREATEST[x_]]], x_] := True

From this, one immediately derives Theorem GT–FU–2 as a corollary:

SubstTest[implies, and[subclass[u, v], subclass[v, w]], subclass[u, w], v -> composite[GREATEST[x], inverse[GREATEST[x]]], v -> intersection[x, inverse[x]], w -> Id]

Out[7] = or[FUNCTION[GREATEST[x]], not[subclass[intersection[x, inverse[x]], Id]]] = True

In[8]:= or[FUNCTION[GREATEST[x_]], not[subclass[intersection[x_, inverse[x_]], Id]]] := True

The results for LEAST[x] follow immediately by replacing x with inverse[x]. (This result did not appear in the Otter work.)

SubstTest[subclass, composite[GREATEST[y], inverse[GREATEST[y]]], intersection[y, inverse[y]], y -> inverse[x]]

Out[9] = subclass[composite[LEAST[x], inverse[LEAST[x]]], x] = True

In[10]:= subclass[composite[LEAST[x_], inverse[LEAST[x_]]], x_] := True

The function rule also generalizes, yielding Theorem LT–FU:

SubstTest[implies, subclass[intersection[y, inverse[y]], Id], FUNCTION[GREATEST[y]], y -> inverse[x]]

Out[11] = or[FUNCTION[LEAST[x]], not[subclass[intersection[x, inverse[x]], Id]]] = True

In[12]:= or[FUNCTION[LEAST[x_]], not[subclass[intersection[x_, inverse[x_]], Id]]] := True