the successor ordinals form a proper class

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This notebook contains an elementary derivation of the fact that the successor ordinals form a proper class. An automated proof of this theorem was obtained in 1999 using McCune’s program Otter. It is Corollary ON-SC-4B in my paper on computer proofs in ordinal number theory.

Some of the same ideas in this proof can be used to derive the fact that the limit ordinals also form a proper class. The latter theorem is slightly more involved in that it requires the use of iteration, and for this reason that derivation will be posted in a separate notebook.

A key fact that is used in both derivations is the sum class axiom: if \( x \) is a set, then so is the sum class \( U[x] \).

The power class axiom implies that if \( x \) is a set, then so is the power class \( P[x] \). The sum class and the power class are related to each other:

The converse of the sum class axiom therefore also holds: \( U[x] \) is a set if and only if \( x \) is a set. The GOEDEL program contains a rewrite rule based on this fact:
The successor function $SUCC$ takes each set $x$ to its successor $\text{succ}[x] = \text{union}[x, \text{singleton}[x]]$. The class of successor ordinals is the intersection of the class $\Omega$ of all ordinals and the class $\text{range}[SUCC]$ of all successors. This intersection can be rewritten as the class of all non-limit ordinals:

$$\text{In}[6] = \text{ImageComp}[SUCC, \text{inverse}[SUCC], \Omega]$$

$$\text{Out}[6] = \text{intersection}[\Omega, \text{range}[SUCC]] = \text{intersection}[\Omega, \text{complement}[\text{fix}[\text{BIGCUP}]]]$$

$$\text{In}[7] = \text{intersection}[\Omega, \text{range}[SUCC]] := \text{intersection}[\Omega, \text{complement}[\text{fix}[\text{BIGCUP}]]]$$

Adding this rewrite rule helps to increase analogies with similar rules for the set $\omega$ of natural numbers:

$$\text{In}[8] = \text{intersection}[\omega, \text{range}[SUCC]]$$

$$\text{Out}[8] = \text{intersection}[\omega, \text{complement}[\text{singleton}[0]]]$$

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It is convenient to introduce a temporary abbreviation for the restriction of a relation $x$ to the class $\Omega$ of ordinals:

$$\text{In}[9] = x := \text{restrict}[x, \Omega, \Omega]$$

The composite of the restriction of the inverse of the membership relation $E$ and the restriction of the successor function $SUCC$ is the restriction of the inverse of the subclass relation $S$:

$$\text{In}[10] = \text{composite}[x[\text{inverse}[E]], x[SUCC]] := x[\text{inverse}[S]]$$

$$\text{Out}[10] = \text{True}$$

From this, one readily deduces:

$$\text{In}[11] = \text{ImageComp}[x[\text{inverse}[E]], x[SUCC], \Omega] // \text{Reverse}$$

$$\text{Out}[11] = \text{U}[\text{intersection}[\Omega, \text{complement}[\text{fix}[\text{BIGCUP}]]]] = \Omega$$

This fact can be made into a rewrite rule:

$$\text{In}[12] = \text{U}[\text{intersection}[\Omega, \text{complement}[\text{fix}[\text{BIGCUP}]]]] := \Omega$$

The fact that $\Omega$ is a proper class now implies that the class of successor ordinals is a proper class:

$$\text{In}[13] = \text{SubstTest}[\text{member}, \text{U}[x], V, x \rightarrow \text{intersection}[\Omega, \text{complement}[\text{fix}[\text{BIGCUP}]]]] // \text{Reverse}$$

$$\text{Out}[13] = \text{member}[\text{intersection}[\Omega, \text{complement}[\text{fix}[\text{BIGCUP}]]], V] := \text{False}$$

This could be made into a rewrite rule, too. We do so as a temporary measure:

$$\text{In}[14] = \text{member}[\text{intersection}[\Omega, \text{complement}[\text{fix}[\text{BIGCUP}]]], V] := \text{False}$$

There is however a more general rewrite rule which follows immediately from the above rule, namely:
This more general rule will be made permanent, and the temporary rule discarded.

\texttt{In[16]:= member[intersection[\text{OMEGA}, \text{complement[fix[BIGCUP]]}], x\_] := False}