transforms of well-founded relations

Johan G. F. Belinfante
2004 September 23

summary

If \( x \) is a function and \( y \) is a well-founded relation, then \( \text{composite}[\text{inverse}[x],y,x] \) is well-founded. This result can be used to obtain a simple proof of the theorem that \( \text{cross}[x,y] \) is well-founded when \( x \) or \( y \) is well-founded.

derivation

The derivation is especially succinct when wrappers are used to represent the hypotheses.

\[
\text{In[2]} := \text{SubstTest}[\text{implies}, \text{and}[\text{equal}[u, \text{funpart}[x]], \text{equal}[v, \text{wf}[y]]], \text{WELLFOUNDED}[\text{composite}[\text{inverse}[u], v, u]], \{u \rightarrow x, v \rightarrow y\}]
\]

\[
\text{Out[2]} = \text{or}[\text{not}[\text{FUNCTION}[x]], \text{not}[\text{WELLFOUNDED}[y]], \text{WELLFOUNDED}[\text{composite}[\text{inverse}[x], y, x]]] = \text{True}
\]
corollary

\textbf{In [6]}:= \texttt{SubstTest[WELLFOUNDED, composite[inverse[funpart[w]], wf[x], funpart[w]], w \rightarrow \text{FIRST}]}

\texttt{Out[6]}= \texttt{WELLFOUNDED[composite[inverse[\text{FIRST}], wf[x], \text{FIRST}]] := True}

\texttt{In[7]}:= \texttt{WELLFOUNDED[composite[inverse[\text{FIRST}], wf[x_], \text{FIRST}]] := True}

It follows that \texttt{cross[wf[x],V]} is well-founded:

\texttt{In[8]}:= \texttt{cross[wf[x], V]}

\texttt{Out[8]}= \texttt{composite[inverse[\text{FIRST}], wf[x], \text{FIRST}]}

Since any subclass of a well-founded relation is well-founded, it now follows that \texttt{cross[wf[x],y]} is well-founded for all \texttt{y}.

\texttt{In[9]}:= \texttt{subclass[cross[wf[x], y], cross[wf[x], V]]}

\texttt{Out[9]}= \texttt{True}

Similarly, one can show that \texttt{cross[x, wf[y]]} is well-founded using the following fact:

\texttt{In[10]}:= \texttt{SubstTest[WELLFOUNDED, composite[inverse[funpart[w]], wf[x], funpart[w]], w \rightarrow \text{SECOND}]}

\texttt{Out[10]}= \texttt{WELLFOUNDED[composite[inverse[\text{SECOND}], wf[x], \text{SECOND}]] := True}

\texttt{In[11]}:= \texttt{WELLFOUNDED[composite[inverse[\text{SECOND}], wf[x_], \text{SECOND}]] := True}

One can use the following fact to show that \texttt{cross[x,y]} is well-founded if and only if \texttt{cross[y,x]} is well-founded.

\texttt{In[12]}:= \texttt{SubstTest[WELLFOUNDED, composite[inverse[funpart[w]], wf[x], funpart[w]], w \rightarrow \text{SWAP}]}

\texttt{Out[12]}= \texttt{WELLFOUNDED[composite[\text{SWAP}, wf[x], \text{SWAP}]] := True}

\texttt{In[13]}:= \texttt{WELLFOUNDED[composite[\text{SWAP}, wf[x_], \text{SWAP}]] := True}