Explicit Version of the Campbell-Baker-Hausdorff Formula:

Integral Representation for $\ln e^x e^y$

Johan G. F. Belinfante
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332

1. Introduction

In the course of describing Lie group multiplication laws in terms of canonical coordinates, F. Schur [1] in 1889 began an investigation of $\ln e^x e^y$ for non-commuting variables $x$ and $y$. The first few terms of this expansion are

$$\ln e^x e^y = x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]]$$

$$+ \frac{1}{12}[y,[y,x]] - \frac{1}{24}[x,[y,[x,y]]] + \ldots$$

where

$[x,y] = x y - y x.$

This series exhibits the symmetry property

$$\ln e^x e^y = -\ln e^{-y} e^{-x}$$

Work by J. E. Campbell [2], H. F. Baker [3], and F. Hausdorff [4] soon led to a description of the terms linear in $y$ but of arbitrary degree in $x$. Writing $ad x$ for the operator defined by
(ad x)y = [x, y], \quad (4)

these terms are expressible as

\[ \ln e^x e^y = x + \frac{\text{ad} x}{1 - e^{-\text{ad} x}} y + \ldots \] \quad (5)

Methods due to E. Witt [5], W. Specht [6], F. Wever [7] and K. O. Friedrichs (see also W. Magnus [8]) can be used to prove that the formal expansion (1) is a Lie element, that is, a formal power series whose terms involve only various higher-order commutators of x and y. Once this is known, an explicit formula can be obtained by a standard projection, as described, for example, by E. B. Dynkin [9]. We do this below in a way that leads to a formal integral representation exhibiting the symmetry (3). If x and y are elements of a Banach algebra, then the series (1) converges in norm for \( \|x\| + \|y\| < \ln 2 \). [10]

2. An Integral Representation

The ring \( F \) of formal power series in non-commuting variables x and y over a field of characteristic zero is spanned by monomials, that is, products \( u_1 \ldots u_n \) where each \( u_i \) is x or y. The linear operator \( \pi: F \to F \) defined by

\[ \pi(1) = 0 \]

\[ \pi(u_1 \ldots u_n) = \frac{1}{n}(\text{ad} u_1) \ldots (\text{ad} u_{n-1})u_n \]

satisfies \( \pi^2 = \pi \), as one establishes by induction on n [11].
A formal power series $f(x,y)$ is a Lie element if and only if $\pi f = f$. We can write any formal power series $f(x,y)$ without constant term as

$$f(x,y) = \xi(x,y)x + \eta(x,y)y,$$

and then, using

$$\frac{1}{n} \int_0^1 t^{n-1} dt = \frac{1}{n},$$

we can rewrite (6) as a formal integral,

$$\pi(f(x,y)) = \int_0^1 dt \{ \xi(t \text{ad } x, t \text{ad } y)x + \eta(t \text{ad } x, t \text{ad } y)y \}.$$

Here $t$ commutes with both $x$ and $y$ and has only a formal significance, defined by (8).

Introducing

$$\phi(w) = \frac{\ln w}{w-1} = 1 - \frac{(w-1)}{2} + \frac{(w-1)^2}{3} - \ldots,$$

we have

$$\ln e^xe^y = \phi(e^xe^y)(e^xe^y - 1) = \phi(e^xe^y)((e^x - 1) + e^xe^y(1-e^{-y})).$$

(11)

Since

$$\phi(w)w = \frac{w \ln w}{w-1} - \frac{\ln w}{w-1} = \phi(w^{-1}),$$

(12)
then (11) can be written as

\[ \ln e^{x^Y} = \phi(e^{x^Y})(e^x - 1) + \phi(e^{-Y}e^{-x})(1 - e^{-Y}). \]  \quad (13)

Since this is a Lie element, then

\[ \ln e^{x^Y} = \pi(\ln e^{x^Y}) = \lim_{n \to \infty} \frac{1}{n!} \pi(\phi(e^{x^Y})x^n - \phi(e^{-Y}e^{-x})(-y)^n). \]

The terms with \( n > 2 \) do not contribute since, for example,

\((\text{ad } x)x = [x,x] = 0\). Hence

\[ \ln e^{x^Y} = \pi(\phi(e^{x^Y})x + \phi(e^{-Y}e^{-x})y) \]

\[ = \int_0^1 \text{d}t(\phi(e^t \text{ad } x \ e^t \text{ad } y)x + \phi(e^{-t} \text{ad } y \ e^{-t} \text{ad } x)y). \]

(15)

This version of the BCH formula displays \( \ln e^{x^Y} \) explicitly as a Lie element, and has the symmetry (3).

3. Related Formulas and Results

Another method of arriving at the integral representation (15) is to begin by establishing that \( h = h(sx,ty) = \ln e^{sx^ty} \) satisfies the formal differential equations [12]-[16]

\[ \frac{\partial h}{\partial s} = \frac{\text{ad } h}{e^{\text{ad } h} - 1} x \]

\[ \frac{\partial h}{\partial t} = \frac{\text{ad } h}{1 - e^{-\text{ad } h}} y \]

(16)
Since $h$ is a Lie element, then [12]

$$
\text{ad } h(sx, ty) = h(s \text{ ad } x, t \text{ ad } y)
$$

(17)

and the equations (16) can be rewritten as

$$
\frac{\partial}{\partial s} \ln e^{sx + ty} = \phi(e^{s \text{ ad } x}e^{t \text{ ad } y})x
$$

(18)

$$
\frac{\partial}{\partial t} \ln e^{sx + ty} = \phi(e^{-t \text{ ad } y}e^{-s \text{ ad } x})y.
$$

Then, by the chain rule,

$$
\frac{d}{dt} \ln e^{tx + ty} = \phi(e^{t \text{ ad } x}e^{t \text{ ad } y})x + \phi(e^{-t \text{ ad } y}e^{-t \text{ ad } x})y.
$$

(19)

and integrating over $t$ from 0 to 1 yields (15).

We can also obtain a generalization of (5) by integrating the second equation of (18) over $t$ from 0 to 1, setting $s = 1$,

$$
\ln e^{x + y} = x + \int_0^1 dt \phi(e^{-t \text{ ad } y}e^{-t \text{ ad } x})y.
$$

(20)

A modified version of this formula was used by S. Greenspan and R. D. Richtmyer to obtain the first five hundred terms of the series (1) by computer [17].
BIBLIOGRAPHY


9b. _____, "On the Representation of the Series \( \log(e^x e^y) \) for Noncommutative \( x \) and \( y \) by Commutators," Mat. Sbornik 25, 155-162 (1949).


