

Class Notes on Poset Theory

Johan G. Belinfante

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Introduction. These notes were originally prepared in July 1972 as a handout for a class in modern algebra taught at the Carnegie-Mellon University in Pittsburgh, Pennsylvania. A proof of Zorn's lemma from the axiom of choice is given along the lines presented by Paul R. Halmos in his book *Naive Set Theory* and by Kenneth S. Miller in the appendix to his book *Elements of Modern Abstract Algebra*. The pretty theorem on the connection between comparability and covering is original. The notes were revised in October 1984 and were used again in classes taught at the Georgia Institute of Technology in Atlanta, Georgia. The original notes were handwritten because of the many figures and specialized symbols that were required. In March 1993 the notes were typeset using PCT_EX and P_ICT_EX.

Partial Orderings. A relation \leq on a class P is a *partial ordering* if it is reflexive, symmetric and transitive:

$$\begin{array}{ll} \text{Reflexive:} & (\forall x \in P) \quad x \leq x, \\ \text{Antisymmetric:} & (\forall x, y \in P) \quad x \leq y \ \& \ y \leq x \Rightarrow x = y, \\ \text{Transitive:} & (\forall x, y, z \in P) \quad x \leq y \ \& \ y \leq z \Rightarrow x \leq z. \end{array}$$

A *poset* (P, \leq) consists of a set P and a partial ordering \leq on P . We define $x < y$ to mean $x \leq y \ \& \ x \neq y$, and we define $a \geq b$ to mean $b \leq a$. Note that \geq is also a partial ordering relation, but $<$ is not. Although the transitive law does hold for $<$, neither the reflexive law nor the antisymmetric law is true.

BIBLIOGRAPHY. A slightly different approach for the axiomatics is used in the following reference.

N. Bourbaki, *Theory of Sets*, Chapter III.

Examples of Posets. Many examples of posets are already familiar.

EXAMPLE. A *trivial example of a poset is $(S, =)$, where S is any set. (Any set can be partially ordered by equality.)*

EXAMPLE. *Any subset of the real line is partially ordered by the usual \leq relation.*

EXAMPLE. *A given set may be partially ordered in more than one way. For example, any set P of non-negative integers is partially ordered not only by the usual \leq relation, but also by the divisibility relation $|$. If $a, b \in P$, we write $a | b$ if and only if there exists an integer m such that $b = ma$.*

EXAMPLE. *Any collection \mathcal{C} of sets can be partially ordered by inclusion. That is, if \mathcal{C} is a set whose elements are sets, then (\mathcal{C}, \subset) is a poset. A simple case is the power set of a set: $\mathcal{C} = \text{Pow } S = \{A \mid A \subset S\}$.*

EXAMPLE. *The dual of a poset (P, \leq) is the poset (P, \geq) .*

EXAMPLE. If L is any collection of statements, then (L, \Rightarrow) satisfies the reflexive and transitive properties, but may fail to satisfy the antisymmetric law. Two distinct statements $A, B \in L$ may be logically equivalent: $A \Rightarrow B$ and $B \Rightarrow A$.

Chains and Antichains. Elements $x, y \in P$ of a poset (P, \leq) are *comparable*, written $x \preceq y$, if $x \leq y$ or $y \leq x$.

A subset C of a poset (P, \leq) is a *chain* if any pair of elements in C are comparable:

$$(\forall x, y \in C) x \preceq y.$$

Chains are also called *totally ordered* subsets.*

A subset A of a poset (P, \leq) is an *antichain* if no two distinct elements of A are comparable:

$$(\forall x, y \in C) x \preceq y \Rightarrow x = y.$$

EXAMPLE. In any poset (P, \leq) , the empty set \emptyset and every singleton $\{x\}$ is both a chain and an antichain.

EXAMPLE. Any subset of the real line (\mathbb{R}, \leq) , with the usual order, is a chain. In particular, $\omega = \{0, 1, 2, \dots\}$ and $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ are chains.

PROPOSITION. Any subset of a chain is a chain.

PROOF: This is clear. Q.E.D.

Intervals and Half-Intervals. In any poset (P, \leq) we can define certain subsets which we call *intervals* and *half-intervals*. If $a, b \in P$, then the following sets are called *closed half-intervals*:

$$\begin{aligned} [a,) &= \{x \in P \mid a \leq x\}, \\ (, b] &= \{x \in P \mid x \leq b\}. \end{aligned}$$

The closed half-interval $(, b]$ is also called a *weak initial segment*. The sets

$$\begin{aligned} (a,) &= \{x \in P \mid a < x\}, \\ (, b) &= \{x \in P \mid x < b\}. \end{aligned}$$

are called *open half-intervals*.

LEMMA. If (P, \leq) is any poset, then

$$\begin{aligned} (\forall a, b \in P) a \leq b &\Rightarrow (, a] \subset (, b], \\ (\forall a, b \in P) a = b &\Rightarrow (, a] = (, b]. \end{aligned}$$

PROOF: This is an easy exercise which uses all three of the poset axioms. Q.E.D.

* The terms *linearly ordered* and *completely ordered* are used by some authors.

The set of all elements in a poset (P, \leq) which are comparable with a given element $x \in P$ is the union of two half-intervals:

$$\{y \in P \mid x \preceq y\} = (, x] \cup [x,).$$

There are four kinds of intervals, obtained by intersecting various types of half-intervals. Sets of the form

$$[a, b] = [a,) \cap (, b],$$

obtained by intersecting two closed half-intervals are called *closed intervals*. Similarly, sets of the form

$$(a, b) = (a,) \cap (, b),$$

obtained by intersecting two open half-intervals are called *open intervals*. Finally, we define two types of *half-open intervals*

$$[a, b) = [a,) \cap (, b),$$

$$(a, b] = (a,) \cap (, b],$$

by intersecting a closed half-interval and an open half-interval.

LEMMA. For any poset (P, \leq) ,

$$(\forall x \in P) [x, x] = \{x\}.$$

PROOF: Use the antisymmetric law. Q.E.D.

Greatest and Least Elements. An element $g \in S$ belonging to a subset $S \subset P$ of a poset (P, \leq) is a *greatest element* of S if

$$(\forall s \in S) s \leq g.$$

Similarly, $l \in S$ is a *least element* of S if

$$(\forall s \in S) l \leq s.$$

A poset need not have any greatest or least element.*

LEMMA. A subset $S \subset P$ of a poset (P, \leq) can have at most one greatest element, and at most one least element.

PROOF: Use the antisymmetric law. Q.E.D.

Note that g is the greatest element of S if $g \in S$ and $S \subset (, g]$. Similarly l is the least element of S if $l \in S$ and $S \subset [l,)$.

* Some authors prefer to use the terms *top element* or *last element* for what we call a *greatest element*. Similarly, the terms *bottom element* or *first element* are used for a *least element*.

Well-ordered Sets. A poset (W, \leq) is *well-ordered* if every nonempty subset has a least element. Well-ordered posets are totally ordered:

PROPOSITION. *Every well-ordered poset is a chain.*

PROOF: If $x, y \in W$ are elements of a well-ordered poset (W, \leq) , then the set $\{x, y\}$ has a least element. Hence $x \leq y$ or $y \leq x$. That is, any two elements of W are comparable. Q.E.D.

EXAMPLE. *Any finite chain is well-ordered. The natural numbers with the usual ordering is a well-ordered set (ω, \leq) .*

The well-ordering of the natural numbers is equivalent to the principle of complete induction. One can derive the principle of induction from the well-ordering property, and conversely.

Not every infinite chain is well-ordered.

EXAMPLE. *The set \mathbb{R} of real numbers with the usual ordering is a chain, but not a well-ordered set. The same is true for the set \mathbb{Q} of rational numbers.*

For example, in both \mathbb{Q} and \mathbb{R} , the half-interval $(0,)$ has no least element. There is no least positive rational or real number: if p is positive, then so is $p/2$.

Monotone Mappings and Poset Isomorphisms. A mapping $f: P_1 \rightarrow P_2$ from a poset (P_1, \leq) to a poset (P_2, \leq) is said to be *monotone* if

$$(\forall x, y \in P_1) x \leq y \Rightarrow f(x) \leq f(y).$$

Monotone mappings are also known as *poset homomorphisms* or simply *poset morphisms*. It is of interest to investigate which concepts in the theory of partially ordered sets are preserved by monotone mappings. Monotone mappings preserve greatest elements, and chains, for example.

PROPOSITION. *If $f: P_1 \rightarrow P_2$ is a monotone mapping from a poset (P_1, \leq) to a poset (P_2, \leq) , and if $x \in S$ is the greatest element of some subset $S \subset P_1$, then $f(x)$ is the greatest element of the subset*

$$f[S] = \{f(s) \mid s \in S\}.$$

PROOF: This is an easy exercise. Q.E.D.

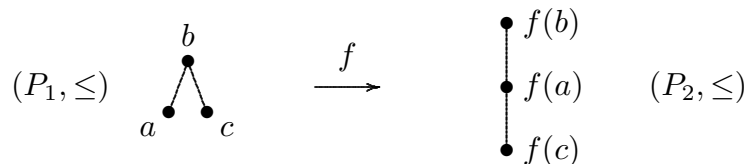
PROPOSITION. *The image of a chain under any monotone mapping is a chain.*

PROOF: This is clear. Q.E.D.

A mapping $f: P_1 \rightarrow P_2$ from a poset (P_1, \leq) to a poset (P_2, \leq) is said to be a *poset isomorphism* if f is one-to-one and onto, and both f and f^{-1} are monotone. Posets (P_1, \leq) and (P_2, \leq) are *isomorphic* if there exists an isomorphism $f: P_1 \rightarrow P_2$.

COUNTEREXAMPLE. A one-to-one and onto monotone mapping need not have a monotone inverse.

CONSTRUCTION: Let (P_1, \leq) and (P_2, \leq) be the posets pictured below, with the mapping f as indicated.



The inverse f^{-1} is not monotone since $f(c) \leq f(a)$ does not imply $c \leq a$. Q.E.F.*

PROPOSITION. For any poset (P, \leq) , the mapping $x \mapsto (, x]$ is a one-to-one monotone mapping from the poset (P, \leq) to the poset $(\text{Pow } P, \subset)$.

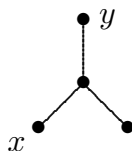
PROOF: See the results of the section on Intervals and Half-Intervals. Q.E.D.

COROLLARY. Any poset (P, \leq) is isomorphic to a poset whose members are sets, partially ordered by inclusion.

Hasse Diagrams and the Covering Relation in a Poset. An element $x \in P$ is covered by an element $y \in P$ in a poset (P, \leq) , written $x \prec y$, if x is strictly less than y , and there is no element strictly between x and y . That is, $x \prec y$ if $x < y$ and $(x, y) = \emptyset$.

EXAMPLE. In the poset of natural numbers (ω, \leq) , we have $(\forall n \in \omega) n \prec n + 1$. In the poset of rational numbers (\mathbb{Q}, \leq) and in the poset of real numbers (\mathbb{R}, \leq) , no element covers any other.

A poset (P, \leq) is finite if the set P has a finite number of elements. Any finite poset (P, \leq) can be represented by a Hasse diagram in which the elements of P are represented as vertices, and the lines represent the covering relation. Note that $x \leq y$ if there is a path leading upward from x to y .



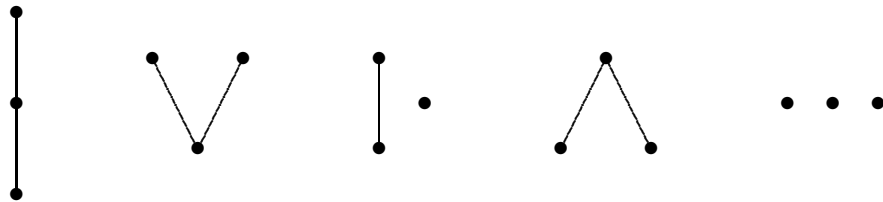
Example of a Hasse Diagram.

* The abbreviation Q.E.F. stands for *quod erat faciendum*. This marks the end of a construction, just as Q.E.D., which stands for *quod erat demonstrandum*, marks the end of a proof.

Examples of Finite Posets. The empty set and any set with a single element can be partially ordered in only one way. A posets with two elements must be isomorphic to one of the two posets whose Hasse diagrams are depicted below. One is a chain, and the other is an antichain.



Show below are Hasse diagrams of the five types of posets with three elements. Note that three of these posets are isomorphic to their own duals; one of these is a chain, and another is an antichain. Of the remaining two, each is isomorphic to the dual of the other. Note also that three of these have connected Hasse diagrams, and the remaining two are made up of disconnected pieces.



There are sixteen posets with 4 elements, of which 10 have connected Hasse diagrams. The number of nonisomorphic posets with n elements grows very rapidly with increasing n .*

n	number of distinct posets	number of connected Hasse diagrams
0	1	1
1	1	1
2	2	1
3	5	3
4	16	10
5	63	44
6	318	238
7	2045	1650
8	16999	14512

Number of Nonisomorphic Posets

* These numbers can be found in a paper by Shawpawm Kumar Das, 1977, *Journal of the Association for Computing Machinery*, Vol. 24, pp. 676–692.

The number of connected Hasse diagrams is related to the total number of posets by a general formula. See for example:

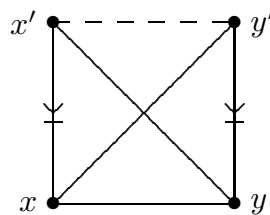
Albert Nijenhuis and Herbert S. Wilf, 1975, *Combinatorial Algorithms*, Academic Press. Cf. p. 79.

Comparability and Covering. We shall now discuss a relation between the concepts of comparability and covering. For this purpose it is convenient to introduce the following notation:

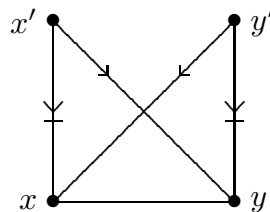
$$\begin{array}{lll}
 x \bullet \text{---} \bullet y & \text{means} & x \preceq y, \\
 x \bullet \text{---} | \text{---} \bullet y & \text{means} & x \prec y, \\
 x \bullet \text{---} \leftarrow \bullet y & \text{means} & x < y, \\
 x \bullet \text{---} \preceq \bullet y & \text{means} & x \leq y.
 \end{array}$$

Dashed lines indicate assertions to be proved.

LEMMA (COMPARABILITY AND COVERING.) *Let x, y, x', y' be elements of a poset (P, \leq) . If $x \prec x'$ and $y \prec y'$, and if $x \preceq y$, $x \preceq y'$ and $x' \preceq y$, then either $x = y$ or $x' \preceq y'$, or both.*



PROOF: If $x' \leq y$, then $x' \leq y < y'$ and hence $x' \preceq y'$, and so we are done in this case. Similarly, by symmetry, we are done if $y' \leq x$. So we may as well assume that both $x < y'$ and $y < x'$. At this point we have the picture:



Because the picture is symmetric, we may also assume without loss of generality that $x \leq y$. We are done if $x = y$, and if $x < y$, then $x < y < x'$, that is, $y \in (x, x')$, violating the assumption that $x \prec x'$. Q.E.D.

Minimal and Maximal Elements. An element $m \in P$ of a poset (P, \leq) is said to be a *maximal element* if there is no greater element in P , that is, if

$$(\forall x \in P) m \leq x \Rightarrow m = x.$$

Similarly, an element $m \in P$ of a poset (P, \leq) is a *minimal element* if there is no lesser element in P .

PROPOSITION. An element $m \in P$ in a poset (P, \leq) is maximal if and only if $[x,) = \{x\}$. Similarly, $m \in P$ is minimal if and only if $(, x] = \{x\}$.

PROOF: This is clear from the definitions. Q.E.D.

PROPOSITION. If a poset has a greatest element, then that element is the one and only maximal element in that poset.

PROOF: Easy exercise. Q.E.D.

EXAMPLE. Minimal and maximal elements need not exist in general. If they do exist, they need not be unique. A maximal element need not be a greatest element.

CONSTRUCTION: If $\omega = \{0, 1, 2, 3, \dots\}$ denotes the system of natural numbers, then the poset (ω, \leq) has a minimal element, but no maximal element. If S is any set, then every element of the poset $(S, =)$ is both minimal and maximal. If S holds more than one element, there is no greatest or least element. Q.E.F.

Maximal elements, unlike greatest elements, do not behave well under monotone mappings, but we do have a weak result:

PROPOSITION. If $f: P_1 \rightarrow P_2$ is a one-to-one and onto monotone mapping from a poset (P_1, \leq) to a poset (P_2, \leq) , and if P_2 has a maximal element, then so does P_1 .

PROOF: If $y = f(x)$ is a maximal element of P_2 , then x is a maximal element of P_1 . Q.E.D.

The above result, in conjunction with Zorn's lemma to be discussed later, could possibly be used to prove existence of maximal elements in certain posets.

Covering Operations. A covering operation f on a poset (P, \leq) is a mapping $f: P \rightarrow P$ for which

$$(\forall x \in P) x \prec f(x).$$

A covering operation need not be monotone.

An example of a covering operation is the *successor operation* $n \mapsto n + 1$ on the poset (ω, \leq) of natural numbers. A nonempty poset with a covering operation must have infinitely many elements; if a is one of them, we have

$$a \prec f(a) \prec f(f(a)) \prec \dots$$

In general, a poset with a covering operation f can not have a maximal element. For if m were a maximal element, then $f(m)$ would be strictly greater than m , which is impossible.

PROPOSITION. If a well-ordered set (W, \leq) has no greatest element, then there is a unique covering operation on W .

PROOF: Let $a \in W$ be any element of a well-ordered set (W, \leq) . If $(a,) = \emptyset$, then a would be the greatest element of W . If W has no greatest element, then the open half-interval $(a,)$ is not empty, and therefore has a least element, say b . Then $a \prec b$, and clearly b is the only element which covers a . The unique covering operation assigns $a \mapsto b$. Q.E.D.

PROPOSITION. If $x \in P$ is any element of a poset (P, \leq) , then the closed half-interval $[x,)$ is invariant under any covering operation on P .

PROOF: If $y \in [x,)$, then $x \leq y \mid\prec f(y)$, and thus $f(y) \in [x,)$. Q.E.D.

Upper and Lower Bounds. An element $b \in P$ is called an *upper bound* for a subset $S \subset P$ of a poset (P, \leq) if

$$(\forall s \in S) s \leq b.$$

Similarly, an element $a \in P$ is called an *lower bound* for $S \subset P$ if

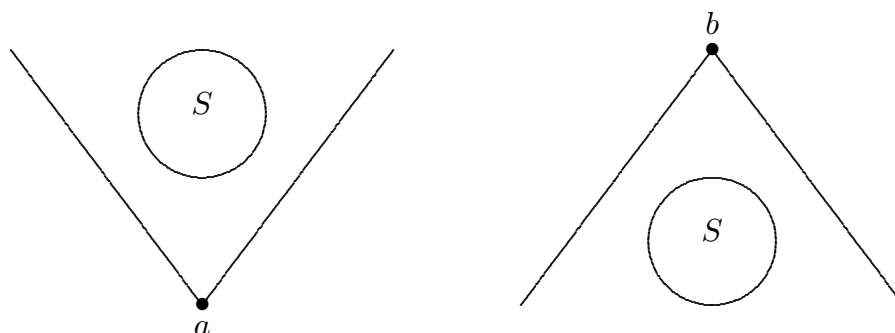
$$(\forall s \in S) a \leq s.$$

These definitions can be restated as follows:

LEMMA. An element b of a poset P is an upper bound for a subset S of P if and only if $S \subset (, b]$. Similarly, an element a in P is a lower bound for a subset S if and only if $S \subset [a,)$.

PROOF: This follows directly from the definitions of upper and lower bounds and the sets $[a,)$ and $(, b]$. Q.E.D.

We sometimes draw pictures of upper and lower bounds as follows:



Pictures of Upper and Lower Bounds.

A subset $S \subset P$ of a poset (P, \leq) is said to be *bounded above* if there exists an upper bound for S in P . Similarly, a subset $S \subset P$ is said to be *bounded below* if there is a lower bound for S in P . A subset $S \subset P$ is *bounded* if it is bounded both above and below.

EXAMPLE. The half-intervals $(, b)$ and $(, b]$ are bounded above. The half intervals $(a,)$ and $[a,)$ are bounded below. The intervals (a, b) , $(a, b]$, $[a, b)$ and $[a, b]$ are all bounded subsets.

The empty set is, of course, a bounded subset of every poset. Also, any singleton is a bounded subset.

COUNTEREXAMPLE. In general, a finite subset of a poset need not be bounded.

CONSTRUCTION: Any subset with two or more elements in the poset $(S, =)$ fails to be bounded. Q.E.F.

Directed Posets. A poset is said to be *directed* if every finite subset is bounded above. (A better term, perhaps, would be *directed upward*. The term *directed downward* could then be applied to the dual concept of a poset in which every finite subset is bounded below.)

PROPOSITION. *A poset is directed if and only if every set of two elements is bounded above. That is, a poset (P, \leq) is directed if and only if for all $x, y \in P$, there exists an element $z \in P$ such that $x \leq z$ and $y \leq z$.*

PROOF: Easy exercise. Q.E.D.

What happens if we require not only that every finite subset is bounded above but that every subset is bounded above? We remark that in this case we have the following simple result.

PROPOSITION. *A poset (P, \leq) has a greatest element if and only if every subset of P is bounded above.*

PROOF: If P itself has an upper bound, then that upper bound must be the greatest element of P . Conversely, if P has a greatest element, then that greatest element is an upper bound for every subset of P . Q.E.D.

COROLLARY. *A finite poset is directed if and only if it has a greatest element.*

Bounded subsets are well-behaved under monotone mappings.

PROPOSITION. *Let $f: P_1 \rightarrow P_2$ be a monotone mapping from a poset P_1 to a poset P_2 . If a subset $S \subset P_1$ is bounded above in P_1 , then its image $f[S] \subset P_2$ under f is bounded above in P_2 .*

PROOF: If b is an upper bound for the subset S , then $f(b)$ is an upper bound for the image $f[S]$. Indeed, since $s \leq b$ for all s in S , then $f(s) \leq f(b)$. Q.E.D.

PROPOSITION. *The image of a directed subset under a monotone mapping is a directed subset.*

PROOF: Let $f: P_1 \rightarrow P_2$ be monotone, and let S be a subset of P_1 . Any two elements of $f[S]$ can be represented as $f(x)$ and $f(y)$ where x and y belong to S . Since S is directed, there exists an element z in S which is an upper bound for both x and y . Since f is monotone, then $f(z)$ is an upper bound for $f(x)$ and $f(y)$. Q.E.D.

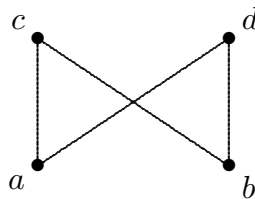
PROPOSITION. *If a directed poset has a maximal element, then that element is the greatest element.*

PROOF: If m is a maximal element, and if x is any element, then since the poset is directed, there is an element z which is an upper bound for both m and x . Since m is maximal, then we must have $m = z$. Hence m is an upper bound for every element x in P . Q.E.D.

Least Upper and Greatest Lower Bounds. Let (P, \leq) be a poset. An element $b \in P$ is the *least upper bound* of a subset $S \subset P$ if b is the least element of the set of all upper bounds for S . If a set $S \subset P$ has a least upper bound, we denote it by $\text{lub } S$ or $\bigvee S$. Similarly, an element is the *greatest lower bound* of a subset S if it is the greatest element of the set of all lower bounds for S . If a set $S \subset P$ has a greatest lower bound, we denote it by $\text{glb } S$ or $\bigwedge S$. In some books, the least upper bound of S is called the *supremum*, denoted by $\text{sup } S$ and the greatest lower bound of S is called the *infimum*, denoted by $\text{inf } S$.

COUNTEREXAMPLE. *A minimal upper bound need not be a least upper bound.*

CONSTRUCTION: In the poset depicted below, elements c and d are both minimal upper bounds for the set $S = \{a, b\}$, but there is no least upper bound. Q.E.F.



PROPOSITION. *If $x \in P$ is any element of a poset (P, \leq) , then*

$$\bigvee(, x] = x.$$

PROOF: This is clear. Q.E.D.

For open half-intervals, the results are more complicated. For example, in the poset (ω, \leq) of the natural numbers, if $n > 0$, then $\bigvee(, n) = n - 1$, whereas in the poset (\mathbb{R}, \leq) , where \mathbb{R} is the set of all real numbers, we have $\bigvee(, x) = x$ for all $x \in \mathbb{R}$. More generally:

PROPOSITION. *If (C, \leq) is a chain in which no element covers another, then $(\forall x \in C) \bigvee(, x) = x$.*

PROOF: Clearly x is an upper bound for the half-interval $(, x)$. If $y \in C$ is another upper bound for the half-interval $(, x)$, then $(, x) \subset (, y]$. Since (C, \leq) is a chain, either $y < x$ or $x \leq y$. If $y < x$, then $(, x) = (, y]$. If $z \in (x, y)$, then $z \in (, x)$ and $z \notin (, y]$, which is absurd. Hence $(x, y) = \emptyset$, and so $x \mid\!< y$, contrary to the hypothesis that no element covers another in (C, \leq) . Hence $x \leq y$. That is, x is the least upper bound of the half-interval $(, x)$. Q.E.D.

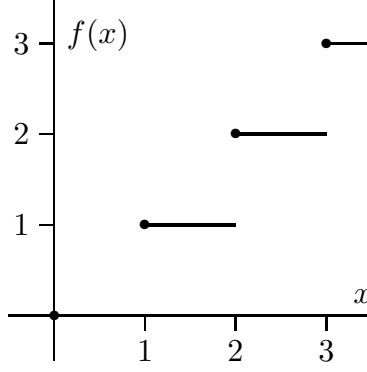
PROPOSITION. *If a subset $S \subset P$ of a poset (P, \leq) has a greatest element g , then $g = \bigvee S$.*

PROOF: Clearly $S \subset (, g]$, so g is an upper bound for S . If b is also an upper bound for S , then since $g \in S$, we must have $g \leq b$. Hence g is the least element of the set of all upper bounds for S . Q.E.D.

COUNTEREXAMPLE. *Monotone mappings need not preserve least upper bounds.*

CONSTRUCTION: Let \mathbb{R} be the set of all real numbers, ordered in the usual way. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the *staircase function* defined by

$$(\forall x \in \mathbb{R}) f(x) = \lfloor x \rfloor = \text{greatest integer } n \text{ such that } n \leq x.$$



If $S = [0, 1)$, then $\bigvee S = 1$ and $f[S] = \{0\}$, so that $\bigvee f[S] = 0$, whereas $f(\bigvee S) = f(1) = 1$. Thus $\bigvee f[S] \neq f(\bigvee S)$. Q.E.F.

Semi-Lattices. A poset (P, \leq) is a *join-semilattice* if for any pair of elements $x, y \in P$, there is a least upper bound in P for the set $\{x, y\}$. We write $x \vee y = \bigvee\{x, y\}$. The symbol \vee is called *join* or *cup*. Similarly, A poset (P, \leq) is a *meet-semilattice* if for any pair of elements $x, y \in P$, there is a greatest lower bound in P for the set $\{x, y\}$. We write $x \wedge y = \bigwedge\{x, y\}$. The symbol \wedge is called *meet* or *cap*.

PROPOSITION. Any join-semilattice is a directed poset.

PROOF: This is clear. Q.E.D.

EXAMPLE. If S is any set, and $\text{Pow } S$ is its power set, then $(\text{Pow } S, \subset)$ is both a join-semilattice and a meet-semilattice, with \vee and \wedge being the set-theoretic union and intersection, respectively.

LEMMA. If (P, \leq) is a join-semilattice, then

$$\begin{aligned} (\forall x, y \in P) \quad x \leq x \vee y \quad \& \quad y \leq x \vee y, \\ (\forall x, y \in P) \quad x \leq z \quad \& \quad y \leq z \quad \Rightarrow \quad x \vee y \leq z. \end{aligned}$$

PROOF: The element $x \vee y$ is the least upper bound of $\{x, y\}$. Q.E.D.

THEOREM. If (P, \leq) is a join-semilattice, then

$$\begin{aligned} \text{Idempotent Law:} \quad & (\forall x \in P) \quad x \vee x = x, \\ \text{Commutative Law:} \quad & (\forall x, y \in P) \quad x \vee y = y \vee x, \\ \text{Associative Law:} \quad & (\forall x, y, z \in P) \quad (x \vee y) \vee z = x \vee (y \vee z). \end{aligned}$$

PROOF: Since x is obviously the greatest element of $\{x, x\}$, then $x \vee x = x$. Since $\{x, y\} = \{y, x\}$, then $x \vee y = y \vee x$. The associative law follows from the lemma. Q.E.D.

Lattices. A poset (L, \leq) is a *lattice* if it is both a join-semilattice and a meet-semilattice. That is, any pair of elements $x, y \in L$ has both a least upper bound $x \vee y \in L$ and a greatest lower bound $x \wedge y \in L$.

COROLLARY. *Any lattice is a directed poset.*

PROPOSITION. *Any chain (C, \leq) is a lattice.*

PROOF: If $x, y \in C$ then either $x \leq y$ or $y \leq x$. If $x \leq y$, then $x \wedge y = x$ and $x \vee y = y$. If $y \leq x$, then $x \wedge y = y$ and $x \vee y = x$. Q.E.D.

We may summarize this as:

well-ordered set \Rightarrow chain \Rightarrow lattice \Rightarrow join-semilattice \Rightarrow directed poset.

THEOREM (ABSORPTIVE LAWS). *If (L, \leq) is a lattice, then*

$$\begin{aligned} (\forall x, y \in L) \quad x \vee (x \wedge y) &= x, \\ (\forall x, y \in L) \quad x \wedge (x \vee y) &= x. \end{aligned}$$

PROOF: Clearly $x \leq x \vee (x \wedge y)$. Since $x \leq x$ and $x \wedge y \leq x$, then x is an upper bound for the set $\{x, x \wedge y\}$. Since $x \vee (x \wedge y)$ is the least upper bound of this same set, then $x \vee (x \wedge y) \leq x$. By the antisymmetric property of partial ordering, it follows that $x \vee (x \wedge y) = x$. The second statement is proved in a similar fashion. Q.E.D.

A comment on axiomatics is in order. We have defined a lattice as a certain type of poset. Alternatively, one can view a lattice as a set L equipped with two binary operations \vee and \wedge subject to six axioms: the two commutative laws, the two associative laws, and the two absorptive laws. One can then reintroduce the partial ordering \leq on L by defining $x \leq y$ to mean $x = x \wedge y$. Note that, on account of the absorptive laws, the statements $x = x \wedge y$ and $x \vee y = y$ are equivalent. Incidentally, the idempotent laws need not be explicitly postulated since they actually follow from the absorptive laws. Indeed:

$$\begin{aligned} x &= x \vee [x \wedge (x \vee x)] = x \vee x, \\ x &= x \wedge [x \vee (x \wedge x)] = x \wedge x. \end{aligned}$$

BIBLIOGRAPHY. There are lots of books on lattices, and some of them treat posets as well, but most deal with special topics.

G. Birkhoff, *Lattice Theory*, First edition: 1940; Second edition: 1948; Third edition: 1961.

G. Szasz, 1963, Academic Press, *Introduction to Lattice Theory*.

D. E. Rutherford, 1965, Hafner, New York, *Introduction to Lattice Theory*.

G. Grätzer, 1971, *Lattice Theory*.

These are just a few of the classics; there are lots more!

A monotone mapping $f: L_1 \rightarrow L_2$ from one lattice to another is called a *lattice homomorphism* if

$$\begin{aligned}(\forall x, y \in L) f(x \vee y) &= f(x) \vee f(y), \\(\forall x, y \in L) f(x \wedge y) &= f(x) \wedge f(y).\end{aligned}$$

We say that f *preserves* the lattice operations \vee and \wedge . In general, a monotone mapping need not preserve the lattice operations.

Complete Lattices. In a lattice, any nonempty finite set $\{x_1, \dots, x_n\}$ has a least upper bound and a greatest lower bound, because

$$\begin{aligned}\bigvee \{x_1, x_2, \dots, x_n\} &= (\dots((x_1 \vee x_2) \vee x_3) \vee \dots \vee x_n), \\ \bigwedge \{x_1, x_2, \dots, x_n\} &= (\dots((x_1 \wedge x_2) \wedge x_3) \wedge \dots \wedge x_n).\end{aligned}$$

The empty set and infinite subsets, however, need not have least upper and greatest lower bounds. For example, the poset (\mathbb{Z}, \leq) of all the integers, with the usual order, is a chain, and therefore a lattice, but neither \emptyset nor \mathbb{Z} has a least upper bound. In the case of \emptyset , every integer is an upper bound, but there is no least integer; in the case of \mathbb{Z} , there is no upper bound at all.

We say that a poset (P, \leq) is a *complete lattice* if every subset $S \subset P$ has a least upper bound $\bigvee S \in P$, and a greatest lower bound $\bigwedge S \in P$. Actually, the existence of greatest lower bounds could be omitted from this definition in view of the following theorem.

THEOREM. *If every subset S of a poset (P, \leq) has a least upper bound $\bigvee S$, then every subset S has a greatest lower bound $\bigwedge S$.*

PROOF: The set L of all lower bounds for S has a least upper bound $\bigvee L \in P$. Any element $s \in S$ is an upper bound for L , and hence $(\forall s \in S) \bigvee L \in P \leq s$. It follows that $\bigvee L \in L$, that is, $\bigvee L$ is the greatest element of L ; that is, $\bigvee L$ is the greatest lower bound for S . Q.E.D.

The converse is also true; if every subset has a greatest lower bound, then every subset has a least upper bound. The proof is similar; the least upper bound of a set S is the greatest lower bound $\bigwedge U$ of the set U of all upper bounds of S .

PROPOSITION. *A complete lattice (L, \leq) has a least element and a greatest element.*

PROOF: Since any element $x \in L$ is an upper bound for \emptyset , the least upper bound $\bigvee \emptyset$ is the least element of L . Similarly, $\bigwedge \emptyset$ is the greatest element of L . Q.E.D.

COROLLARY. *Any finite nonempty lattice is complete.*

PROOF: Any finite nonempty subset of a lattice has a least upper bound and a greatest lower bound. Q.E.D.

PROPOSITION. *For any set S , the poset $(\text{Pow } S, \subset)$ is a complete lattice.*

PROOF: If $A \subset S$, we have $\bigvee A = \bigcup A$. If A is not empty, then $\bigwedge A = \bigcap A$. If $A = \emptyset$, then $\bigwedge A = S$. Q.E.D.

EXAMPLE. Any closed interval $[a, b]$ on the real line (\mathbb{R}, \leq) is a complete lattice if $a < b$. A closed interval $[a, b]$ on the rational line (\mathbb{Q}, \leq) is not a complete lattice if $a < b$.

For example, because $\sqrt{2}$ is irrational, the set

$$\{x \in \mathbb{Q} \mid x > 0 \ \& \ x^2 < 2\} \subset [0, 2]$$

has no least upper bound in the closed interval $[0, 2] \subset \mathbb{Q}$.

A somewhat more interesting example is provided by topology. Recall that a *topology* \mathcal{T} on a set S is a collection of subsets of S , called *open subsets* such that

$$\begin{aligned} S &\in \mathcal{T}, \\ (\forall \mathcal{S} \subset \mathcal{T}) \bigcup \mathcal{S} &\in \mathcal{T}, \\ (\forall G_1, G_2 \in \mathcal{T}) G_1 \cap G_2 &\in \mathcal{T}. \end{aligned}$$

This definition of a topology is essentially equivalent to that given on page 37 in John L. Kelley's book, *General Topology*.

Note that since $\emptyset \subset \mathcal{T}$, then $\emptyset = \bigcup \emptyset \in \mathcal{T}$. The *interior* of any subset $A \subset S$ is defined to be the union of the collection of all open subsets contained in A .

PROPOSITION. If \mathcal{T} is a topology on a set S , then (\mathcal{T}, \subset) is a complete lattice.

PROOF: If $\mathcal{S} \subset \mathcal{T}$, then $\bigcup \mathcal{S} \in \mathcal{T}$ is the least upper bound for \mathcal{S} . Note that if \mathcal{S} is not empty, then the greatest lower bound $\bigwedge \mathcal{S}$ is not the intersection $\bigcap \mathcal{S}$, but rather the interior of the intersection. When $\mathcal{S} = \emptyset$, we have $\bigwedge \mathcal{S} = \bigcup \mathcal{T} = S$. Q.E.D.

The Complete Lattice Fixed Point Theorem. An element $x \in L$ is a *fixed point* for an operation $f: L \rightarrow L$ if $f(x) = x$. A subset $S \subset L$ is *invariant* under an operation f if $f[S] \subset S$; that is, S is invariant under f if

$$(\forall x \in S) f(x) \in S.$$

In particular, x is a fixed point of f if and only if the singleton $\{x\}$ is invariant under f .

THEOREM. Any monotone operation $f: L \rightarrow L$ on a complete lattice (L, \leq) has a fixed point.

PROOF: The subset $S = \{x \in L \mid x \leq f(x)\}$ is invariant under f because

$$x \leq f(x) \implies f(x) \leq f(f(x)).$$

Since L is a complete lattice, then $\bigvee S \in L$ and $(\forall x \in S) x \leq \bigvee S$, so that $x \leq f(x) \leq f(\bigvee S)$. Hence $f(\bigvee S)$ is an upper bound for S , and $\bigvee S \leq f(\bigvee S)$. Therefore $\bigvee S \in S$. Since S is invariant, then $f(\bigvee S) \in S$. Hence $f(\bigvee S) \leq \bigvee S$. It follows that $\bigvee S$ is a fixed point of f . Q.E.D.

Davis proves that the converse is also true.

A. C. Davis, 1955, Pacific J. Math. **5**, 311–319, *A Characterization of Complete Lattices*.

Another fixed point theorem (due to Bourbaki) will be proved later; the Bourbaki fixed point theorem uses Zorn's lemma.

Filters and Ideals in Lattices. A subset $S \subset L$ of a lattice (L, \leq) is a *sublattice* if

$$(\forall x, y \in S) x \wedge y \in S,$$

$$(\forall x, y \in S) x \vee y \in S.$$

PROPOSITION. *The intersection of any nonempty collection of sublattices of a lattice is a sublattice.*

PROOF: This is elementary. Q.E.D.

A subset $I \subset L$ of a lattice (L, \leq) is an *ideal* if

$$(\forall x \in I)(\forall y \in L) x \wedge y \in I,$$

$$(\forall x, y \in I) x \vee y \in I.$$

An ideal is clearly a sublattice, but a sublattice need not be an ideal.

EXAMPLE. *The empty set \emptyset and L itself are ideals of any lattice (L, \leq) .*

The ideals \emptyset and L are called *improper ideals* of L .

EXAMPLE. *If $x \in L$ is any element of a lattice (L, \leq) , then the closed half-interval $(, x]$ is an ideal of L .*

The ideals $(, x]$ are called *principal ideals* of L .

Ideals can be characterized in another way which does not make use of the lattice operations. A subset $H \subset P$ of a poset (P, \leq) is *hereditary* if

$$(\forall h \in H) (, h] \subset H.$$

That is, $H \subset P$ is hereditary if and only if

$$(\forall x \in P)(\forall h \in H) x \leq h \Rightarrow x \in H.$$

Recall also that a subset $D \subset P$ of a poset (P, \leq) is directed if

$$(\forall x, y \in D)(\exists z \in D) x \leq z \ \& \ y \leq z.$$

PROPOSITION. *A subset $I \subset L$ of a lattice (L, \leq) is an ideal of L if and only if I is directed and hereditary.*

PROOF: We leave the proof as an easy exercise. Q.E.D.

The concept of a hereditary directed subset makes sense in any poset, thereby generalizing the concept of an ideal in a lattice. This concept is interesting mainly in infinite posets in view of the following theorem.

PROPOSITION. Every nonempty hereditary directed subset of a finite poset (P, \leq) is a closed half-interval $(, x]$.

PROOF: Any finite directed subset has a greatest element. If a hereditary subset H has a greatest element x , then $H = (, x]$. Q.E.D.

PROPOSITION. The set of all ideals of a lattice (L, \leq) , partially ordered by inclusion, is a complete lattice.

PROOF: If \mathcal{C} is a nonempty collection of ideals, then $\bigcap \mathcal{C}$ is an ideal and this ideal is the greatest lower bound for \mathcal{C} . Also $\bigwedge \emptyset = L$. Q.E.D.

Note that if \mathcal{C} is a collection of ideals, then the least upper bound $\bigvee \mathcal{C}$ need not be the union $\bigcup \mathcal{C}$, but instead $\bigvee \mathcal{C}$ is the intersection of the set of all ideals which contains the union.

The mapping $x \mapsto (, x]$ is a one-to-one monotone mapping which allows us to embed any lattice (L, \leq) in its lattice of ideals. Thus, any lattice can be embedded in a complete lattice. (Is this mapping a lattice homomorphism?)

By a *complete chain* we mean a chain which is a complete lattice. For example, any closed interval on the real line $[a, b] \subset \mathbb{R}$ is a complete chain.

PROPOSITION. The lattice of ideals of a chain (C, \leq) is a complete chain.

PROOF: If I and J are ideals of C , and if I is not contained in J , then there exists an element $x \in I$ which does not belong to J . If $y \in J$, then $(, y] \subset J$ and since $x \not\leq y$ and $x \notin J$, we must have $y < x$. Since y is an arbitrary element of J , then $J \subset (, x) \subset I$. Q.E.D.

COROLLARY. Every chain can be monotonically embedded in a complete chain.

PROOF: Use the mapping $x \mapsto (, x]$. Q.E.D.

A comment is in order here. Starting with the rational line (\mathbb{Q}, \leq) , this completion procedure does not yield the real line (\mathbb{R}, \leq) . Indeed, the real line fails to be a complete chain because it lacks a least and greatest element.

A subset D of a lattice (L, \leq) is a *dual ideal* if

$$\begin{aligned} (\forall x \in D)(\forall y \in L) x \vee y \in D, \\ (\forall x, y \in D) x \wedge y \in D. \end{aligned}$$

An dual ideal is clearly a sublattice. A proper dual ideal in a lattice L is called a *filter*. The filters of a lattice are partially ordered by inclusion. A maximal filter is called an *ultrafilter*.

Various Chain Conditions. We shall say that a subset $S \subset P$ of a poset (P, \leq) is *inductive* if every nonempty chain $C \subset S$ has an upper bound in S . (This is precisely the hypothesis of Zorn's lemma.) We say that a subset $S \subset P$ of a poset (P, \leq) is *strictly inductive* if every nonempty chain $C \subset S$ has a least upper bound in P , and $\bigvee C \in S$. Finally, we say that a subset $S \subset P$ of a poset (P, \leq) is *Noetherian* or satisfies the *ascending chain condition* (often abbreviated as *a.c.c.* if every nonempty chain $C \subset S$ has a greatest element. The following implications are clear:

$$\text{finite} \Rightarrow \text{Noetherian} \Rightarrow \text{strictly inductive} \Rightarrow \text{inductive}.$$

Note that the empty set satisfies all three conditions.

The above terminology for chain conditions on a poset is not completely standard. The term *inductive* is used with the present meaning by Bourbaki in *Theory of Sets*, Chapter III, Section 2.4. However, the term *inductive set* is also used in ordinal number theory with a different meaning which need not concern us here. The term *strictly inductive* is my own invention. The term *Noetherian* is used in the present sense by Bourbaki in *Theory of Sets*, Chapter III, Section 6.5. The same term is used with a similar meaning in ring theory; an abelian ring is *Noetherian* if its lattice of ideals satisfies the ascending chain condition.

Any poset with a greatest element, of course, is inductive.

PROPOSITION. *A chain is inductive if and only if it has a greatest element.*

PROOF: If (C, \leq) is an inductive chain, then C itself must have an upper bound $b \in C$. Since $x \leq b$ for all $x \in C$, the element b is the greatest element of C . Q.E.D.

The connection between ordinary mathematical induction and inductive sets is slightly topsy turvy. The set of positive integers, of course, is not even an inductive set because it is a chain with no greatest element. We need to turn the integers upside down:

EXAMPLE. *The set of negative integers, with the usual ordering, is an example of an infinite Noetherian poset.*

We say that a poset (P, \leq) satisfies the *descending chain condition*, abbreviated *d.c.c.*, if every nonempty chain $C \subset P$ has a least element.

EXAMPLE. *A well-ordered set is a chain which satisfies the d.c.c.*

EXAMPLE. *Any complete lattice is strictly inductive; for example, any closed interval $[a, b] \subset \mathbb{R}$ with $a < b$ is strictly inductive. A closed interval of the rational numbers, with the usual ordering, is an example of an inductive poset which is not strictly inductive.*

COUNTEREXAMPLE. *A subset of an inductive set need not be inductive.*

CONSTRUCTION: The closed interval $[0, 1] \subset \mathbb{R}$ is inductive, but the subset

$$\{0, 1/2, 3/4, 7/8, \dots\} \subset [0, 1]$$

is a chain with no greatest element, and is therefore not inductive. Q.E.F.

The importance of the concept of a strictly inductive poset is due to the following:

THEOREM. *The set $\text{Ch } P$ of all chains in a poset (P, \leq) , partially ordered by inclusion, is a strictly inductive poset.*

PROOF: Suppose \mathcal{C} is a nonempty chain of chains; that is, \mathcal{C} is a nonempty chain in the poset $(\text{Ch } P, \subset)$. If $x, y \in \bigcup \mathcal{C}$, then there exist chains $C_1, C_2 \in \mathcal{C}$ with $x \in C_1$ and $y \in C_2$. Since \mathcal{C} is a chain, either $C_1 \subset C_2$ or $C_2 \subset C_1$. In either case, both x and y belong to the bigger one of these chains. Hence $x \leq y$, and it follows that $\bigcup \mathcal{C}$ is a chain in P . Clearly $\bigcup \mathcal{C}$ is the least upper bound for the chain $\mathcal{C} \subset \text{Ch } P$. Q.E.D.

PROPOSITION. *If a poset (P, \leq) is strictly inductive, and if $x \in P$, then the subsets $(, x]$ and $[x,)$ are strictly inductive.*

PROOF: We leave this as an exercise. Q.E.D.

PROPOSITION. *The intersection of any nonempty collection of strictly inductive subsets is strictly inductive.*

PROOF: We leave this as an exercise. Q.E.D.

Spinal Elements. The following terminology is not standard, but will prove useful in connection with Zorn's lemma, to be discussed later. We shall say that an element $x \in P$ of a poset (P, \leq) is a *spinal element* if it is comparable with every element in P . The *spine* of a poset (P, \leq) is the set of all spinal elements; we denote it by

$$\text{Spine } P = \{x \in P \mid (\forall y \in P) x \leq y\}.$$

The spine of any poset is a chain, possibly empty. Note that $x \in \text{Spine } P$ if and only if $P = (, x] \cup [x,)$.

EXAMPLE. *If a poset (P, \leq) has a least or greatest element, then these are spinal elements.*

EXAMPLE. *Any chain (C, \leq) is its own spine.*

We shall prove a theorem about the spine of a strictly inductive poset. For this we need the following lemma.

LEMMA. *If (P, \leq) is a strictly inductive poset, and if $x \in P$, then $(, x] \cup [x,)$ is a strictly inductive subset.*

PROOF: Any nonempty chain $C \subset (, x] \cup [x,)$ is the union of two chains $C = C_+ \cup C_-$ where $C_+ \subset [x,)$ and $C_- \subset (, x]$. If C_+ is not empty, then $\bigvee C = \bigvee C_+ \in [x,)$. If $C_+ = \emptyset$, then $\bigvee C = \bigvee C_- \in (, x]$. Q.E.D.

THEOREM. *The spine of a strictly inductive poset is either empty or has a greatest element.*

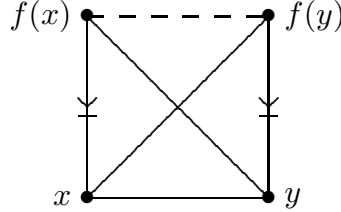
PROOF: If (P, \leq) is a strictly inductive poset, and if its spine $S = \text{Spine } P$ is not empty, then $\bigvee S \in P$. If $x \in P$, then $S \subset (, x] \cup [x,)$, and hence $\bigvee S \in (, x] \cup [x,)$. Hence

$$(\forall x \in P) x \leq \bigvee S.$$

Hence $\bigvee S \in S$. Q.E.D.

A Theorem due to Zermelo. In this section we shall prove a theorem due to Zermelo which is a close relative of Zorn's lemma, but which does not require the axiom of choice. We begin with a technical lemma which will be needed later.

LEMMA. *If f is a covering operation on a poset (P, \leq) , and if $x \in P$ is a spinal element of P , then the set of elements comparable to $f(x)$ is invariant under f .*



PROOF: We have to show that

$$(\forall y \in P) y \leq f(x) \Rightarrow f(y) \leq f(x).$$

Since x is a spinal element, this follows from the diagram. Q.E.D.

A strictly inductive subset of a poset which is invariant under a covering operation f is called an f -tower. We will prove theorem due to Zermelo which asserts that f -towers do not exist, (except in the trivial case of the empty set.) "Towers topple." For this we need the following lemma.

LEMMA. *Let f be a covering operation on a poset (P, \leq) . The intersection of any non-empty set of f -towers is itself an f -tower.*

PROOF: Invariant subsets and strictly inductive sets are preserved under intersections. Q.E.D.

THEOREM (ZERMELO). *A nonempty strictly inductive poset (P, \leq) has no covering operations.*

PROOF: Suppose f were a covering operation on (P, \leq) , and $b \in P$. Then $[b,)$ is an f -tower, and $b \in [b,)$. The intersection M of all f -towers to which b belongs is an f -tower, and $b \in M$. This minimal f -tower is contained in any f -tower to which b belongs, so $M \subset [b,)$. Clearly b is the least element of M , so the spine S of M is not empty. We proved earlier that the spine of a non-empty strictly inductive poset has a greatest element. Let $g \in S$ be the greatest element of the spine S . By a technical lemma proved earlier, the set $T \subset M$ of all elements in M which are comparable to $f(g)$ is invariant under f . Note that $b \in T$ since $b \leq g \leq f(g)$. Since M and $(, f(g)] \cup [f(g),)$ are both strictly inductive, so is their intersection T . Thus T is an f -tower to which b belongs. Since M is the smallest f -tower to which b belongs, then $M \subset T$. Since $T \subset M$ and $M \subset T$, then $T = M$. Hence $f(g)$ is a spinal element of M . This is absurd because g is the greatest spinal element of M and $f(g)$ is strictly greater than g . Q.E.D.

Zorn's Lemma. Up to now, we have not made use of the axiom of choice; Zermelo's theorem on the nonexistence of covering operations on a strictly inductive poset does not require the axiom of choice. From it we can deduce a more useful result, called *Zorn's lemma*, whose proof does make essential use of the axiom of choice.

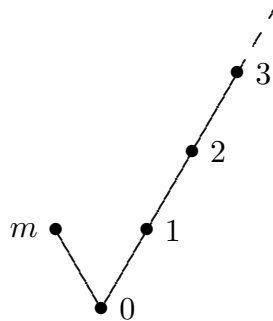
THEOREM (ZORN'S LEMMA). *Every nonempty inductive poset has a maximal element.*

PROOF: Suppose (P, \leq) is a nonempty inductive poset which has no maximal element. If $C \subset P$ is a chain in P , then by the hypothesis that P is inductive, there exists an upper bound for C , say x . (Note that if C is the empty set, then any element of P is an upper bound for C .) Since x is not a maximal element of P , there exists a strictly larger element, say y . Since $C \subset (, x]$ and $x < y$, then $y \notin C$. Therefore $C \cup \{y\}$ is a chain. Clearly the chain C is covered by the chain $C \cup \{y\}$ in the poset $(\text{Ch } P, \subset)$ of all chains in P . By the axiom of choice, there is a function $f: \text{Ch } P \rightarrow \text{Ch } P$ which assigns to each chain $C \in \text{Ch } P$ some such chain $f(C) = C \cup \{y\}$ which covers C . Then f is a covering operation on the poset of chains in P . Since the poset $(\text{Ch } P, \subset)$ is strictly inductive, by Zermelo's theorem, there can be no covering operation on $(\text{Ch } P, \subset)$. This is a contradiction. Q.E.D.

The converse of Zorn's lemma is not valid.

COUNTEREXAMPLE. *A poset (P, \leq) having a maximal element need not be inductive.*

CONSTRUCTION: In the poset shown below, the element m is maximal, but the chain $\{0, 1, 2, \dots\}$ has no upper bound. Q.E.D.



COROLLARY. *Any chain in a poset is contained in a maximal chain.*

PROOF: The poset $(\text{Ch } P, \subset)$ of all chains in a poset (P, \leq) is strictly inductive. It is not empty since $\emptyset \in \text{Ch } P$. If $C \in \text{Ch } P$, then the half-interval $[C,) \subset \text{Ch } P$ is a strictly inductive subset of $\text{Ch } P$. Hence there is a maximal element in $[C,)$. Q.E.D.

COROLLARY. *Any antichain in a poset is contained in a maximal antichain.*

PROOF: The union of any chain of antichains is an antichain. Hence the poset of all antichains in a given poset is strictly inductive. Q.E.D.

COROLLARY (BOURBAKI FIXED POINT THEOREM.) *Any operation $f: P \rightarrow P$ on a nonempty inductive poset P satisfying $(\forall x \in P) x \leq f(x)$ has a fixed point.*

PROOF: If x is a maximal element of P , then $x \leq f(x)$ implies that $f(x) = x$. Q.E.D.