This is a take home midterm. You can use your notes, my online notes on canvas and the textbooks book. You are supposed to work on your own text without external help. I'll be available to answer question in person or via email. Please, write clearly and legibly and take a readable scan before uploading.

To solve the Exam problems, I have not collaborated with anyone nor sought external help and the material presented is the result of my own work.

Name (print): ______________________________

Question:	1	2	3	4	5	Total
Points:	10	20	30	40	20	120
Score:						

Question 1 . . . . *10 point*

Let X be a standard normal random variable. Compute all moments of X, that is compute $m_k = \mathbb{E}(X^k)$ for every $k \geq 0$. (**Hint:** you can use induction. Alternatively you can use the m.g.f..)

Solution: We need to compute

$$\mathbb{E}(X^k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^k e^{-\frac{x^2}{2}} \, dx \, .$$

Observe that

$$-xe^{-\frac{x^2}{2}} = \frac{d}{dx} e^{-\frac{x^2}{2}}$$

so that integrating by part we get

$$\int_{-\infty}^{\infty} x^k e^{-\frac{x^2}{2}} \, dx = -x^{k-1} e^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} + (k-1) \int_{-\infty}^{\infty} x^{k-2} e^{-\frac{x^2}{2}} \, dx$$

so that

$$m_k = (k-1) m_{k-2} \, .$$

Clearly $m_k = 0$ if k is odd while we can see by induction that

$$m_{2k} = \prod_{n=1}^{k} (2n-1) = \frac{(2n)!}{2^n n!} \, .$$

Alternatively we have

$$m_k = \frac{d^k}{dx^k} M_X(t) = \frac{d^k}{dx^k} e^{\frac{t^2}{2}} \, .$$

Observe that

$$e^{\frac{t^2}{2}} = \sum_{n=0}^{\infty} \frac{t^{2n}}{2^n n!}$$

from which we get the same conclusion as above.

Question 2 . . . . *20 point*

Let X_t describe the price of a stock at time t, where time is measured in days and $t = 0, 1, 2, \ldots$. Assume that X_t satisfies:

$$X_{t+1} = \Delta_t X_t$$

where Δ_t, $t = 0, 1, 2, \ldots$, form a family of i.i.d. random variables with p.d.f.:

$$f_\Delta(\delta) = \frac{1}{\delta\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\log\delta - \mu)^2}{2\sigma^2}\right) \qquad \delta \geq 0\,.$$

(a) (10 points) Assume that $X_0 = 1$ with probability 1. Find the p.d.f. of X_t for $t > 0$. (**Hint:** you can express Δ is term of a normal r.v..)

Solution: Call $Z_t = \log\Delta_t$. From the formula of change of variable we get that the p.d.f. of Z is

$$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(z - \mu)^2}{2\sigma^2}\right)$$

that is Z is normal with $E(Z) = \mu$ and $var(Z) = \sigma^2$. Thus we can write

$$X_{t+1} = e^{Z_t} X_t$$

from which we get $X_t = e^{T_t}$ where $T_t = \sum_{s=0} t - 1 Z_s$ is a normal r.v. with $\mathbb{E}(T_t) = t\mu$ and $var(T_t) = t\sigma^2$. Changing variables back we get:

$$f_{X_t}(x) = \frac{1}{x\sqrt{2\pi t\sigma^2}} \exp\left(-\frac{(\log x - t\mu)^2}{2t\sigma^2}\right) \qquad x \geq 0\,.$$

(b) (10 points) Assume now that $\mu = 0.1$ and $\sigma^2 = 0.2$. Find $\bar{x}$ such that

$$\mathbb{P}(X_{10} > \bar{x}) = 0.75.$$

You can use a calculator like the one here.

Solution:
We have

$$\mathbb{P}(X_{10} > \bar{x}) = \mathbb{P}(T_{10} > \log(\bar{x})) = \mathbb{P}\left(\frac{T_{10} - 10\mu}{10\sigma^2} > \frac{\log(\bar{x}) - 10\mu}{10\sigma^2}\right) =$$
$$1 - \Phi\left(\frac{\log(\bar{x}) - 10\mu}{10\sigma^2}\right) = 0.75\,.$$

Since $\Phi^{-1}(0.25) = -0.674$ we get

$$\bar{x} = \exp(-2 \cdot 0.674 + 1) = 0.706\,.$$

Question 3 .... *30 point*

Let X, Y have a joint distribution that is uniform on the unit circle in $\mathbb{R}^2$, that is

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2+y^2 \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

(a) (10 points) Let $R = X^2 + Y^2$ and Θ be the r.v. defined by

$$\cos(\Theta) = \frac{X}{\sqrt{X^2+Y^2}} \qquad \sin(\Theta) = \frac{Y}{\sqrt{X^2+Y^2}}$$

with $0 \leq \Theta < 2\pi$. Show that R and Θ are in dependent and find the p.d.f. f_R of R and f_Θ of Θ.

Solution: By construction we have

$$X = \sqrt{R}\cos(\Theta) \qquad Y = \sqrt{R}\sin(\Theta)$$

so that the Jacobian is

$$J = \begin{pmatrix} -\frac{\cos(\theta)}{2\sqrt{r}} & -\sqrt{r}\sin(\theta) \\ -\frac{\sin(\theta)}{2\sqrt{r}} & \sqrt{r}\cos(\theta) \end{pmatrix}$$

and $|J| = 1/2$. Thus we get $f_{R,\Theta} = 1/2\pi$ for $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Thus R is uniform in $[0,1]$ while Θ is uniform in $[0, 2\pi]$ and they are independent.

(b) (10 points) Let $T = \sqrt{-2\log(X^2+Y^2)}$. Fint the p.d.f. f_T of T.

Solution: We have

$$\mathbb{P}(T \geq t) = \mathbb{P}(X^2+Y^2 \leq e^{-t^2/2}) = e^{-t^2/2}$$

so that

$$f_T(t) = te^{-t^2/2}.$$

(c) (10 points) Consider the r.v.

$$U = \frac{X}{\sqrt{X^2+Y^2}}\sqrt{-2\log(X^2+Y^2)}$$
$$V = \frac{Y}{\sqrt{X^2+Y^2}}\sqrt{-2\log(X^2+Y^2)}\,.$$

Show that U, V are i.i.d normal standard r.v.

Solution: We have

$$U = \cos(\Theta)T$$
$$V = \sin(\Theta)T\,.$$

or

$$T = \sqrt{U^2+V^2}$$
$$\Theta = \arctan\left(\frac{U}{V}\right)$$

and we get $|J| = 1/\sqrt{U^2+V^2}$. Using the change of variable formula we get

$$f_{U,V}(u,v) = \frac{1}{2\pi}e^{-(u^2+v^2)/2} = \frac{1}{\sqrt{2\pi}}e^{-u^2/2}\frac{1}{\sqrt{2\pi}}e^{-v^2/2}\,.$$

Question 4 .... *40 point*

Let X_i, $i = 1, 2, 3, \ldots$, be a family of i.i.d random variables with exponential distribution with parameter 1, that is

$$f_{X_i}(x) = e^{-x}$$

For $n \geq 1$ defines $T_n = \sum_{i=1}^{n} X_i$.

You can think that X_i is the time between the i-th and the $i+1$-th arrival of a request for service at a certain computer server. In this case, T_n is the time of arrival of the n-th request for service.

(a) (10 points) Find the p.d.f. of T_n. (**Hint**: Use change of variables to find the joint p.d.f. of the T_i, $i = 1, 2, \ldots, n$, and then compute the marginal on T_n. Alternatively you can use induction.)

Solution: The joint p.d.f of $X_1, X_2, \ldots X_n$ is $f_{X,n}(x_1, \ldots, x_n) = \exp(-\sum_{i=1}^{n} x_i)$ for $x_i > 0$ and 0 otherwise. Using the formula of change of variables we get that the joint p.d.f. of $T_1, T_2, \ldots, T_n$ is

$$f_{T,n}(t_1, \ldots, t_n) = \exp(-t_n)$$

if $0 < t_1 < t_2 < \cdots < t_n$ and 0 otherwise. Thus we get

$$f_{T_n}(t_n) = \int_{0<t_1<\cdots<t_n} dt_1 \cdots dt_{n-1} e^{-t_n} = \frac{t_n^{n-1}}{(n-1)!} e^{-t_n} .$$

(b) (10 points) Let N_t be the number of arrivals before time t, that is

$$N_t = \max\{n \,|\, T_n < t\}\,.$$

Show that N_t is a Poisson r.v. with expected value t.

Solution: Observe that $\mathbb{P}(N_t < n) = \mathbb{P}(T_n \geq t)$ so that

$$\mathbb{P}(N_t \leq n) = \int_t^\infty \frac{s^n}{n!} e^{-s} ds = \int_0^\infty \frac{(t+s)^n}{n!} e^{-(t+s)} ds = \sum_{k=0}^{n} \binom{n}{k} \frac{t^k e^{-t}}{n!} \int_0^\infty s^{n-k} e^{-s} ds = \sum_{k=0}^{n} \frac{t^k e^{-t}}{k!}$$

so that

$$\mathbb{P}(N_t = n) = \mathbb{P}(N_t \leq n) - \mathbb{P}(N_t \leq n-1) = \frac{t^n e^{-t}}{n!}$$

(c) (20 points) (Bonus) For $t_2 > t_1$ let $N_{t_1,t_2} = N_{t_2} - N_{t_1}$. Show that N_{t_1,t_2} is a Poisson r.v. with expected value $t_2 - t_1$ and that $N_{t_1,t_2} \perp N_{t_3,t_4}$ if $(t_1,t_2) \cap (t_3,t_4) = \emptyset$.

Solution: From point a) we get that, given $k > 0$

$$\begin{aligned}\mathbb{P}(N_{t_1} =& m \,\&\, N_{t_2} \geq m+k) = \mathbb{P}(T_m < t_1 \,\&\, T_{m+1} > t_1 \,\&\, T_{m+k} < t_2) = \\ &\int_{0<s_1<\cdots<s_m<t_1<s_{m+1}<\cdots<s_{m+k}<t_2} ds_1 \cdots ds_{m+k} e^{-s_{m+k}} = \\ &\frac{t_1^m}{m!}\int_{t_1<s_{m+1}<\cdots<s_{m+k}<t_2} ds_{m+1} \cdots ds_{m+k} e^{-s_{m+k}} = \\ &\frac{t_1^m}{m!}\int_{0<\tau_1<\cdots<\tau_k<t_2-t_1} d\tau_1 \cdots d\tau_k e^{-\tau_k - t_1} = \\ &\mathbb{P}(N_{t_1} = m)\mathbb{P}(N_{t_1-t_2} \geq k)\,.\end{aligned}$$

It follows that

$$\mathbb{P}(N_{t_1} = m \,\&\, N_{t_2} = m+k) = \mathbb{P}(N_{t_1} = m)\mathbb{P}(N_{t_2-t_1} = k)$$

and finally

$$\begin{aligned}\mathbb{P}(N_{t_2} - N_{t_1} = k) =& \sum_m \mathbb{P}(N_{t_1} = m \,\&\, N_{t_2} = m+k) = \\ &\sum_m \mathbb{P}(N_{t_1} = m)\mathbb{P}(N_{t_2-t_1} = k) = \mathbb{P}(N_{t_2-t_1} = k)\end{aligned}$$

With a very similar computation we get, for $t_1 < t_2 < t_3$,

$$\begin{aligned}\mathbb{P}(N_{t_1} = m \,\&\, N_{t_2} = m+k \,\&\, N_{t_3} = m+k+q) = \\ \mathbb{P}(N_{t_1} = m)\mathbb{P}(N_{t_2-t_1} = k)\mathbb{P}(N_{t_3-t_2} = q)\end{aligned}$$

that, summing over k, gives $N_{t_1} \perp N_{t_2,t_3}$.

Question 5 .... *20 point*

Two question on continuous/discrete r.v.

(a) (10 points) Let X be continuous r.v. uniformly distributed in $[0, 1]$. Consider the r.v.

$$Z = \max\{X, 0.5\}\,.$$

Find the c.d.f. of Z. Is Z a continuous r.v.? Is it discrete?

Solution: Clearly we have $\mathbb{P}(Z < 0.5) = 0$ while $\mathbb{P}(Z \leq z) = \mathbb{P}(X \leq z)$ if $z \geq 0.5$ so that we have

$$F_Z(z) = \begin{cases} 0 & z < 0.5 \\ z & 0.5 \leq z < 1 \\ 1 & z \geq 1 \end{cases}$$

Z is not a continuous r.v. since F_Z is not continuous and it is not discrete since F_Z is not piecewise constant.

(b) (10 points) Let X be continuous r.v. with p.d.f. f_X and Y be a discrete r.v. with p.m.f. p_Y. Moreover X and Y are independent. Find the p.d.f. of $Z = X + Y$. Is Z a continuous r.v.?

Solution: We have

$$\mathbb{P}(Z \leq z) = \sum_y \mathbb{P}(X \leq z - y \,\&\, Y = y) = \sum_y \mathbb{P}(X \leq z - y)\mathbb{P}(Y = y)$$

so that

$$f_Z(z) = \sum_y f_X(z - y)p_Y(y)$$

In general we cannot say whether it is continuous but if Y takes only finitely many values than Z is continuous.