1.2. Continuation of solutions.

It is convenient to let the pair $(x,\, I)$ denote a solution of the initial value problem (1.2) on the interval I. We say that a solution $(y,\, J)$ is an *extension* of $(x,\, I)$ if $I \subset J$ and $x(t) = y(t)$ on I. We say that $(x,\, I)$ is a *maximal solution* of (1.2) if and only if, for any extension $(y,\, J)$, we have $J = I$. In this case, we refer to I as the *maximal interval of existence* of the solution of (1.2). If (x, I), $I = (\alpha,\, \beta)$, is a solution of (1.2) and ∂D denotes the boundary of D, then we say that $(t,\, x(t))$ *approaches the boundary of* D, denoted by $(t,\, x(t)) \to \partial D$, as $t \to \alpha$ (resp. β) in I if, for any compact set $K \subset D$, there exists $\epsilon > 0$ such that $(t, x(t)) \in D \setminus K$ for all $t \in (\alpha,\, \alpha + \epsilon)$ (resp. $t \in (\beta - \epsilon,\, \beta)$). We emphasize that the definition of $(t,\, x(t))$ approaches the boundary of D implies that the solution must eventually leave any compact set of D.

Theorem 2.1. *For any solution $(x^*,\, I^*)$ of (1.2), there exists a maximal solution $(x,\, I)$ that is an extension of $(x^*,\, I^*)$. The interval I is open, $I = (\alpha,\, \beta), -\infty \leq \alpha,\, \beta \leq +\infty$, and $(t, x(t)) \to \partial D$ as $t \to$ either end point of I.*

Proof. We let $S = \{(y,\, J) : (y,\, J) \text{ is an extension of } (x^*,\, I^*)\,\}$ and define the partial ordering on S by $(y_1,\, J_1) \leq (y_2,\, J_2)$ if and only if $J_1 \subset J_2$ and $y_1(t) = y_2(t)$ on J_1. It is clear that S is not empty. If S_0 is a totally ordered subset of S, we define $I = \cup\{\, J : (y,\, J) \in S_0\,\}$ and $x(t) = y(t)$ for all $t \in I$. It follows that $(x,\, I)$ is an upper bound of S_0 and is clearly an extension of $(x^*,\, I^*)$. From Zorn's Lemma, we conclude that $(x^*,\, I^*)$ is contained in the maximal solution $(x,\, I)$.

We prove that I is open by contradiction. Suppose that the end points of I are $\alpha < \beta$. If I contains β and $x(\beta) = \zeta$, then $(\beta,\, \zeta) \in D$ and the local existence theorem implies that there is a solution y of (1.1) with initial state ζ at time β existing on an interval $J = [\beta - \epsilon,\, \beta + \epsilon]$ for some $\epsilon > 0$. If we define $z(t) = x(t)$ on I, $z(t) = y(t)$ on $(\beta, \beta + \epsilon)$, then $(z,\, I \cup (\beta,\, \beta + \epsilon))$ is an extension of $(x,\, I)$, which contradicts the fact that (x, I) is a maximal solution. The endpoint α is treated in the same way.

We now show that $(t,\, x(t)) \to \partial D$ as $t \to \beta$. It is clear that we only need to consider $\beta < \infty$. If the maximal solution $(x,\, I)$ does not approach the boundary, then there is a compact set $K \subset D$ and a monotone sequence $\{\, t_n\,\}, t_n \to \beta$ as $n \to \infty$, such that $(t_n,\, x(t_n)) \in K$ for all n. The compactness of K implies that $\{\, x(t_n)\,\}$ has a convergent subsequence which is again denoted by $\{\, x(t_n)\,\}$. Assume that $x(t_n) \to \zeta$ as $n \to \infty$. If we show that $x(t) \to \zeta$ as $t \to \beta$, then we can apply Theorem 1.1 to extend the solution to $(\alpha,\, \beta + \epsilon)$ for some $\epsilon > 0$, which contradicts the fact that $(x,\, I)$ is a maximal solution.

It remains to show that $x(t) \to \zeta$ as $t \to \beta$. If $\bar{a}, \bar{b}$ are positive real numbers chosen so that $\Gamma = \{\,(t,\, x) : |t - \beta| \leq \bar{a},\; |x - \zeta| \leq \bar{b}\,\} \subset D$, then we let $M = \sup\{|f(t, x)| : (t, x) \in \Gamma\}$. We choose $0 < a \leq \bar{a}$, $0 < b \leq \bar{b}$ such that $aM \leq b$ and define $R = \{(t, x) : |t - \beta| \leq a, |x - \zeta| \leq b\}$, $R^* = \{(t, x) : |t - \beta| \leq a, |x - \zeta| \leq \frac{1}{2}\, b\}$. Since $(t_n, x(t_n)) \to (\beta, \zeta)$, we can choose n large enough so that $(t_n, x(t_n)) \in R^*$ and $|t_n - t_{n+1}| < \frac{b}{2M}$. For $t_n \leq t \leq t_{n+1}$, we claim that $(t, x(t)) \in R$. If t^* is the first time that $(t, x(t))$ leaves R and remains in Γ, then $|x(t^*) - \zeta| > b$. On the other hand,

$$|x(t^*) - x(t_n)| \leq \left| \int_{t_n}^{t^*} f(s, x(s)) ds \right| \leq M|t^* - t_n| \leq M|t_{n+1} - t_n| < \frac{b}{2}.$$

and

$$|x(t^*) - \zeta| \leq |x(t^*) - x(t_n)| + |x(t_n) - \zeta| < \frac{b}{2} + \frac{b}{2} = b.$$

Therefore, $(t, x(t)) \in R$ for all $t \geq t_n$. Now

$$\begin{aligned} |x(t) - \zeta| \leq |x(t) - x(t_n)| + |x(t_n) - \zeta| &\leq \left| \int_{t_n}^{t} f(s, x(s)) ds \right| + |x(t_n) - \zeta| \\ &\leq M|t - t_n| + |x(t_n) - \zeta|, \end{aligned}$$

and so $x(t) \to \zeta$ as $t \to \beta$. This completes the proof of the theorem.

Remark 2.1. In the applications, the region D oftenb is given as $D = \mathbb{R} \times \mathbb{R}^d$. Let $B_r(0) = \{x : |x| < r\}$. Suppose that we have obtained in some way the following apriori information: if $\xi \in B_r(0)$, then there is an $r_1 < r$ such that the solution $x(t, 0, \xi) \in B_{r_1}(0)$ for all t in its maximal interval of existence. For any fixed $T > 0$, let $D_r = (-1, T) \times B_r(0)$. The continuation theorem implies that $x(t, 0, \xi) \to \partial D_r$ as $t \to T$. Since $r_1 < r$, we conclude that $\lim_{t \to T} x(t, 0, \xi)$ exists and belongs to $B_r(0)$. Thus, the solution can be extended beyond T and must exist on $[0, \infty)$.

As an example, consider $\dot{x} = -x^3$. If ξ is given, then the solution must always satisfy $|x(t, 0, \xi)| \leq |\xi|$.

Exercise 2.1. Give an example of a scalar differential equation $\dot{x} = f(t, x)$ with the following property: there is a sequence of maximal solutions $(x_n, I_n), n = 0, 1, 2, \ldots$, such that $I_n = (-\infty, \infty), n = 1, 2, \ldots$, $I_0 = (-1, 1)$ and $x_n(0) \to x_0(0)$ as $n \to \infty$.

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