1. Let $f$ be the function defined by

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

a) Find the partial derivatives of $f$ at $(0, 0)$ or explain carefully why they do not exist.

$$\frac{\partial f}{\partial x}(0, 0) \overset{h \to 0}{=} \lim \left( \frac{f(0 + h) - f(0, 0)}{h} \right) = \lim \left( \frac{0 - 0}{h} \right) = 0.$$ 

From the symmetry of $f$, we see that $\frac{\partial f}{\partial y}(0, 0) = 0$, also.

b) Are the partial derivatives continuous at $(0, 0)$? Explain.

For $(x, y) \neq (0, 0)$, we have

$$\frac{\partial f}{\partial x}(x, y) = \frac{(x^2 + y^2)y - xy2x}{(x^2 + y^2)^2} = \frac{y^3 - x^2y}{(x^2 + y^2)^2}.$$ 

To check for continuity at $(0, 0)$, we need to see if

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial x}(0, 0) = 0$$

Note that

$$\frac{\partial f}{\partial x}(0, y) = \frac{y^3}{y^4} = \frac{1}{y},$$

so that $\frac{\partial f}{\partial x}(x, y)$ can be made as large as you wish by taking $(0, y)$ close enough to $(0, 0)$. Thus $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x, y)$ does not exist, and so $\frac{\partial f}{\partial x}(x, y)$ is not continuous at $(0, 0)$. A very similar argument will show that $\frac{\partial f}{\partial y}(x, y)$ is also not continuous at $(0, 0)$.

2. In the picture below, the curve is a level curve of the function $f(x, y)$, and one of the vectors shown
is the gradient of $f$. The directional derivative of $f$ in the direction of the vector $C$ is negative.

a) Which of the vectors is the gradient of $f$? Explain.

The gradient must be normal to the level curve, so it must be either $B$ or $G$. The directional derivative in the direction of $C$ is the scalar product of the gradient with a vector having the direction of $C$. This is negative only if the cosine of the angle between the two is negative, which happens only if the angle is greater than $\pi/2$. Thus $G$ is the gradient.

b) Which is larger, the directional derivative of $f$ in the direction of $A$ or the directional derivative in the direction of $F$? Explain.

Again, the directional derivative in a direction $u$ is the scalar product of the gradient with a vector in the direction $u$, and this will be bigger where the cosine of the angle between the two is bigger, or where the angle is smaller. Clearly the angle between the gradient $G$ and the vector $F$ is smaller than the angle between $G$ and $A$. Thus the directional derivative in the direction of $F$ is larger than the directional derivative in the direction of $A$.