Suppose you wish to fly (in an airplane) from point A to point B. In the absence of wind, then you simply fly in the direction of the displacement vector \( \mathbf{d} \) from A to B. If the speed of the plane is \( s \) miles per hour (m.p.h.), then the trip will take \( \frac{|\mathbf{d}|}{s} \) hours. More formally, if the velocity of the plane is \( \mathbf{v} \), then the displacement after time \( t \) is simply \( t \mathbf{v} \), and we want to have \( t \mathbf{v} = \mathbf{d} \). Thus, \( t = \frac{|\mathbf{d}|}{|\mathbf{v}|} = \frac{|\mathbf{d}|}{s} \), and \( \mathbf{v} = \frac{1}{t} \mathbf{d} \).

This is pretty simple stuff. The excitement begins when the wind begins to blow. If there is a wind with velocity \( \mathbf{w} \), then the displacement of our plane is no longer \( t \mathbf{v} \), where \( \mathbf{v} \) is the plane’s velocity, but it is \( t(\mathbf{v} + \mathbf{w}) \). Now we need to find \( t \) and \( \mathbf{v} \) so that \( t(\mathbf{v} + \mathbf{w}) = \mathbf{d} \). This vector equation is a bit more difficult to solve. Let’s see what we can do.

First, look at the scalar product of \( \mathbf{v} \) with the equation \( t(\mathbf{v} + \mathbf{w}) = \mathbf{d} \). This gives us

\[
\langle \mathbf{v} \cdot \mathbf{v} + t\mathbf{w} \cdot \mathbf{v} = \mathbf{d} \cdot \mathbf{v} \rangle.
\]

Next, look at the scalar product of \( \mathbf{w} \) with the equation \( t(\mathbf{v} + \mathbf{w}) = \mathbf{d} \):

\[
\langle \mathbf{v} \cdot \mathbf{w} + t\mathbf{w} \cdot \mathbf{w} = \mathbf{d} \cdot \mathbf{w} \rangle.
\]

Substituting the value of \( \mathbf{v} \cdot \mathbf{w} \) from this equation into the first equation gives us

\[
\langle \mathbf{v} \cdot \mathbf{v} + \mathbf{d} \cdot \mathbf{w} - t\mathbf{w} \cdot \mathbf{w} = \mathbf{d} \cdot \mathbf{v} \rangle.
\]

Multiply this one by \( t \) :

\[
t^2\mathbf{v} \cdot \mathbf{v} + t\mathbf{d} \cdot \mathbf{w} - t^2\mathbf{w} \cdot \mathbf{w} = t\mathbf{d} \cdot \mathbf{v}.
\]

Now scalar multiply the very first equation by \( \mathbf{d} \):

\[
\langle \mathbf{v} \cdot \mathbf{d} + t\mathbf{w} \cdot \mathbf{d} = \mathbf{d} \cdot \mathbf{d} \rangle, \text{ or }
\]

\[
\langle \mathbf{v} \cdot \mathbf{d} = \mathbf{d} \cdot \mathbf{d} - t\mathbf{w} \cdot \mathbf{d} \rangle
\]

Finally, substitute this into the previous equation:

\[
t^2\mathbf{v} \cdot \mathbf{v} + t\mathbf{d} \cdot \mathbf{w} - t^2\mathbf{w} \cdot \mathbf{w} = \mathbf{d} \cdot \mathbf{d} - t\mathbf{w} \cdot \mathbf{d}.
\]

Tidying this up, we have
\[ t^2(s^2 - w^2) + 2td \cdot w - d^2 = 0, \]

where \( s = |v| \) is the speed of the plane, \( w = |w| \) is the wind speed, and \( d = |d| \) is the distance between \( A \) and \( B \). Now the possible solutions for this quadratic will be

\[ t = \frac{-d \cdot w \pm \sqrt{(d \cdot w)^2 + d^2(s^2 - w^2)}}{s^2 - w^2}. \]

Let’s reflect on this. First, suppose \( s > w \); that is, suppose the speed of our aircraft is greater than the wind speed. Then \( s^2 - w^2 > 0 \), and so the radical term \( \sqrt{(d \cdot w)^2 + d^2(s^2 - w^2)} > |d \cdot w| \), giving us exactly one positive solution

\[ t = \frac{-d \cdot w + \sqrt{(d \cdot w)^2 + d^2(s^2 - w^2)}}{s^2 - w^2}. \]

Then it’s easy to find \( v = \frac{1}{t}d - w \), and we are all done.

Let’s look at an example. Suppose you are to fly from Atlanta to Charlotte, your plane has a speed of 135 m.p.h., and there is a 35 m.p.h. wind from the Southeast. In this case \( d = |d| = 224 \) miles; \( s = 135 \), \( w = 35 \) m.p.h., and the compass heading from Atlanta to Charlotte is 63 degrees. The angle \( \varphi \) between \( d \) and \( w \) is thus \( \varphi = 45 + 63 = 108 \) degrees. Then

\[ d \cdot w = dw \cos \varphi = (224)(35)(\cos((108/180)\pi)) = -2422.7. \]

Thus

\[ t = \frac{-d \cdot w + \sqrt{(d \cdot w)^2 + d^2(s^2 - w^2)}}{s^2 - w^2} = 1.87 \text{ hours} \]

For the heading, we have

\[ v = \frac{1}{t}d - w = \frac{1}{1.87}(224 \cos(27^\circ)i + 224 \sin(27^\circ)j) - (-35 \cos(45^\circ)i + 35 \sin(45^\circ)j) \]

\[ v = 131.46i + 29.633j \]

Now then \( \arctan(29.633/131.46) = 0.22171 \), or 12.703 degrees. The heading is thus \( 90 - 12.703 = 77.297 \) degrees.

Much more exciting is the case in which the our plane’s speed is less than the wind speed: \( s^2 - w^2 < 0 \). Then the expression under the radical looks like
\[(d \cdot w)^2 + d^2(s^2 - w^2) = d^2w^2 \cos^2 \varphi - d^2(w^2 - s^2) = d^2(w^2 \cos^2 \varphi - w^2 + s^2),\]

where \(\varphi\) is the angle between \(d\) and \(w\). Thus there are real solutions only in case

\[w^2 \cos^2 \varphi - w^2 + s^2 \geq 0, \text{ or} \]
\[\cos^2 \varphi \geq 1 - \left(\frac{s}{w}\right)^2.
\]

Next, we need to decide which are positive. Convince yourself that there will be no positive value of \(t\) if \(d \cdot w < 0\). Thus we must have \(\cos \varphi \geq 0\). Putting this together with our previous result, we need to have

\[\cos \varphi \geq \sqrt{1 - \left(\frac{s}{w}\right)^2}.
\]

Observe that when this is so, there are two positive values of \(t\), giving us two different directions that will take us from \(A\) to \(B\). Can you explain what’s going on?

Let’s try another example. Suppose once again we are to fly from Atlanta to Charlotte in our 135 m.p.h. plane, but now there is a 165 m.p.h. wind from the South. As in the previous example, \(d = 224\) and \(s = 135\). But now, \(w = 165\) and the angle \(\varphi\) between \(d\) and \(w\) is \(\varphi = 63\) degrees. Then

\[\varphi = \arccos\left(\sqrt{1 - \left(\frac{s}{w}\right)^2}\right) = \arccos(0.57496) = 57.5\text{ degrees}.
\]

We see then that it is impossible for us to get to Charlotte in this wind.

Suppose the 165 m.p.h. wind shifts so that it is coming from the Southeast. Then \(\varphi = 63 - 45 = 18\) degrees, and we should be able to make it now. Let’s see how.

\[d \cdot w = (224)(165) \cos(18(\pi/180)) = 35151\]

\[\therefore 35151.
\]

and so

\[t = \frac{-d \cdot w \pm \sqrt{(d \cdot w)^2 + d^2(s^2 - w^2)}}{s^2 - w^2} = \frac{-35151 \pm 28000}{-9000}\]
We see that we have two positive values of $t$:

$$t_1 = \frac{-35151 + 28000}{-9000} = 0.8 \text{ hours, and}$$

$$t_2 = \frac{-35151 - 28000}{-9000} = 7.0 \text{ hours.}$$

Look first at the quick trip, the one which takes a mere 0.8 hours.

$$v = \frac{1}{0.8} \mathbf{d} - \mathbf{w}$$

$$= \frac{1}{0.8} (224 \cos(27^\circ) \mathbf{i} + 224 \sin(27^\circ) \mathbf{j}) - 165(\cos(45^\circ) \mathbf{i} + \sin(45^\circ) \mathbf{j})$$

or,

$$v = 132.8 \mathbf{i} + 10.45 \mathbf{j}$$

Now, $\arctan(10.45/132.8) = 4.5$ degrees. Our heading should thus be $90 - 4.5 = 85.5$ degrees.

Let’s see what the other possibility looks like. Here we have $t = 7.0$ and so

$$v = \frac{1}{7.0} \mathbf{d} - \mathbf{w}$$

$$= \frac{1}{7.0} (224 \cos(27^\circ) \mathbf{i} + 224 \sin(27^\circ) \mathbf{j}) - 165(\cos(45^\circ) \mathbf{i} + \sin(45^\circ) \mathbf{j})$$

$$v = -88.16 \mathbf{i} - 102.14 \mathbf{j}$$

Now, $\arctan(88.16/102.14) = 40.798$, and so the heading is $180 + 40.8 = 220.8$ degrees. Reflect and meditate on these results.

George Cain & Gunter Meyer
School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332-0160