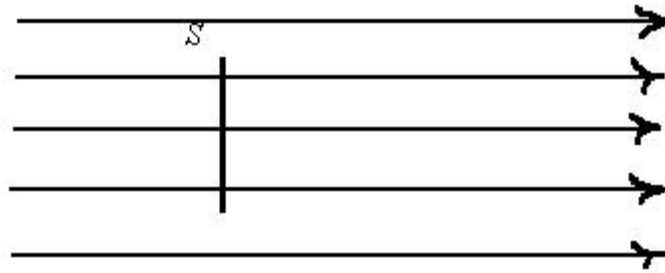


Chapter Sixteen
Integrating Vector Functions

16.1 Introduction

Suppose water (or some other incompressible fluid) flows at a constant velocity \mathbf{v} in space (through a pipe, for instance), and we wish to know the rate at which the water flows across a rectangular surface S that is normal to the stream lines:



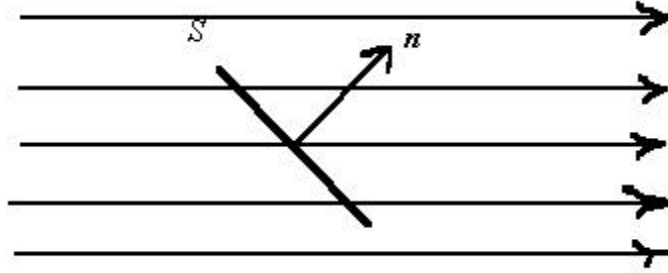
What is the rate at which the fluid flows through S ? Let $M(t)$ denote the total volume of fluid that has passed through the surface at time t . The amount of fluid that flows through during the time between t and $t + \Delta t$ is simply

$$M(t + \Delta t) - M(t) = |\mathbf{v}|a \Delta t ,$$

where a is the area of S . Thus, the rate of flow through S is $\frac{dM}{dt} = |\mathbf{v}|a$.

The result is slightly more complicated when various exciting changes are made. Clearly there is nothing special about the surface's being a rectangle. But suppose that S is placed at an angle to the stream lines instead of being placed normal to the them. Then

we have $\frac{dM}{dt} = \mathbf{v} \cdot \mathbf{n} a$, where \mathbf{n} is a unit normal to the surface S .



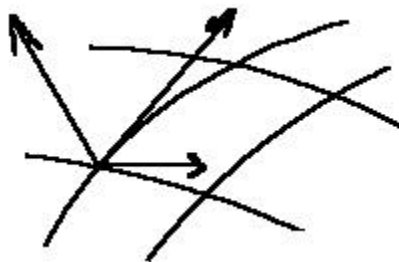
Observe that matters which unit normal to the plane surface we choose. If we choose the other normal ($-n$), then our rate will be the negative of this one. We must thus specify an *orientation* of the surface. We are computing the rate of flow from one side of the surface to the other, and so we have to specify the "sides", so to speak.

16.2 Flux

Now, let's look at the general situation. The surface is not restricted to being a plane surface, and the velocity of the flow is not restricted to being constant in space; it may vary with position as well as time. Specifically, suppose S is a surface, together with an orientation—that is, some means of specifying two "sides"—and suppose $F(r)$ is a function $F: \mathbf{R}^3 \rightarrow \mathbf{R}^3$, which is the velocity of the incompressible fluid. How do we find the rate of flow through the surface S from one side to the other?

First, let's come to grips with the problem of specifying an *orientation* for S . We say that an *orientation* for S is a continuous function $n: S \rightarrow \mathbf{R}^3$ such that $n(r)$ is normal to S and $|n(r)| = 1$ for all $r \in S$. A surface together with an orientation is called an *oriented surface*. At first blush this looks simple enough, and the unsophisticated might guess that every surface has an orientation (or may be *oriented*, as we sometimes say). But this is not so! There are many surfaces for which an orientation does not exist. You may recall from grammar school a simple example of such a surface, the so-called Möbius band, or strip. Here is my feeble attempt to draw one:

Now we see about finding the rate of flow through the oriented surface S . The strategy should be old-hat by now. We subdivide S and look at "small" parallelograms tangent to the surface:



As we have done so often, we suppose the subdivisions are small and approximate the rate of flow, or *flux*, through the subdivision by the rate of flow through the tangent parallelogram.

$$S_i = \mathbf{F}(\mathbf{r}_i^*) \cdot \mathbf{n} A_i,$$

and then add them to obtain yet another type of Riemann sum $R = \sum_{i=1}^n \mathbf{F}(\mathbf{r}_i^*) \cdot \mathbf{n} A_i$. If

these sums have a limiting value as the size of the subdivisions go to zero, this is what we call the integral of \mathbf{F} over the oriented surface S :

$$\int_S \mathbf{F}(\mathbf{r}) \cdot d\mathbf{S}.$$

It should be clear now what we do to evaluate such an integral. As usual, we consider a vector description of the surface S : $\mathbf{r}: D \rightarrow \mathbf{R}^3$, where $D \subset \mathbf{R}^2$. We subdivide S by subdividing the region D into rectangles formed by lines $s = \text{constant}$ and $t = \text{constant}$, and looking at the curves $\mathbf{r}(s, \bar{t})$ and $\mathbf{r}(\bar{s}, t)$ on the surface, exactly as we did in integrating a scalar function over a surface S . Most conveniently now, the vector

product $\frac{\mathbf{r}}{s} \times \frac{\mathbf{r}}{t}$ gives us not only a vector such that $\left| \frac{\mathbf{r}}{s} \times \frac{\mathbf{r}}{t} \right| s t$ is the area of the approximating parallelogram, but also one which is normal to the surface. There is the slight problem of the orientation of S . Thus $\frac{\mathbf{r}}{s} \times \frac{\mathbf{r}}{t}$ may not point in the direction of the specified orientation, in which case, of course, we simply replace $\frac{\mathbf{r}}{s} \times \frac{\mathbf{r}}{t}$ by its negative, $-\frac{\mathbf{r}}{s} \times \frac{\mathbf{r}}{t}$. (We may think of just reversing the roles of s and t .) We have in the Riemann sums,

$$R = \sum_{i=1}^n \mathbf{F}(\mathbf{r}_i^*) \left(\frac{\mathbf{r}}{s} \times \frac{\mathbf{r}}{t} \right) s_i t_i,$$

and, as before, we obtain

$$\int_S \mathbf{F}(\mathbf{r}) dS = \int_D \mathbf{F}(\mathbf{r}(s,t)) \left(\frac{\mathbf{r}}{s} \times \frac{\mathbf{r}}{t} \right) dA.$$

The concept we have developed here is purely mathematical and is done independent of any physical interpretation, such as our fluid flow interpretation. What we have is just an integral of a vector function \mathbf{F} (or field) over an oriented surface S . This is generally called the **flux** of \mathbf{F} over S . There are many physical interpretations of this concept; you have perhaps seen some of them in elementary school physics. There is electric flux, the flux of an electric field; magnetic flux; gravitational flux, *etc., etc.*

Example

Let S be the sphere of radius a oriented so that the normal points "out" of the sphere, and let $\mathbf{F}(\mathbf{r}) = \frac{c}{|\mathbf{r}|^3} \mathbf{r}$, where c is a constant. Let's find $\int_S \mathbf{F}(\mathbf{r}) dS$. Use the vector description of S we used in the first Example of the previous section:

$$\mathbf{r}(s, t) = a \cos s \sin t \mathbf{i} + a \sin s \sin t \mathbf{j} + a \cos t \mathbf{k},$$

$0 \leq s \leq 2\pi$, $0 \leq t \leq \pi$. We have already found that

$$\frac{\mathbf{r}}{s} \times \frac{\mathbf{r}}{t} = a^2 \sin t [-\cos s \sin t \mathbf{i} - \sin s \sin t \mathbf{j} - \cos t \mathbf{k}].$$

Modest meditation should convince you that this normal points into the sphere, and is thus the negative of the one we need for the specified orientation of S .

Next, the integrand is given by

$$\mathbf{F}(\mathbf{r}) = \frac{c}{|\mathbf{r}|^3} \mathbf{r} = \frac{c}{a^3} a [\cos s \sin t \mathbf{i} + \sin s \sin t \mathbf{j} + \cos t \mathbf{k}],$$

and our integral becomes

$$\begin{aligned} \int_S \mathbf{F}(\mathbf{r}) \, d\mathbf{S} &= \int_0^{2\pi} \int_0^\pi \frac{c}{a^2} [\cos s \sin t \mathbf{i} + \sin s \sin t \mathbf{j} + \cos t \mathbf{k}] a^2 \sin t [\cos s \sin t \mathbf{i} + \sin s \sin t \mathbf{j} + \cos t \mathbf{k}] \, ds \, dt \\ &= c \int_0^{2\pi} \int_0^\pi \sin t [\cos^2 s \sin^2 t + \sin^2 s \sin^2 t + \cos^2 t] \, ds \, dt \\ &= c \int_0^{2\pi} \sin t \, ds \, dt = 2c \int_0^\pi \sin t \, dt = 4c. \end{aligned}$$

Note that the radius a of the sphere has disappeared—the value of the integral is independent of the radius of the sphere.

Exercises

1. Find $\int_S [z\mathbf{i} + x^2\mathbf{k}] dS$, where S is that part of the surface $z = x^2 + y^2$ that lies above the square $\{(x, y): -1 \leq x \leq 1, \text{ and } -1 \leq y \leq 1\}$, oriented so that the normal points upward.
2. Find the flux of $\mathbf{F}(x, y, z) = x\mathbf{i} + z\mathbf{j}$ out of the tetrahedron bounded by the coordinate planes and the plane $x + 2y + 3z = 6$.
3. Find the flux of $\mathbf{F}(\mathbf{r}) = \frac{c}{|\mathbf{r}|^3}\mathbf{r}$ out of the surface of the cube $-a \leq x, y, z \leq a$, where c and a are positive constants.
4. Find the flux of the function $\mathbf{F}(x, y, z) = 4x\mathbf{i} + 4y\mathbf{j} + 2z\mathbf{k}$ outward through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$.
5. Find the flux of the function $\mathbf{F}(x, y, z) = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ upward through the surface cut from the cylinder $z = 4 - y^2$ by the planes $x = 0$, $x = 1$, and $z = 0$.
6. Let S be the surface defined by

$$y = \log x, \quad 1 \leq x \leq e, \quad 0 \leq z \leq 1,$$

and let \mathbf{n} be the orientation of S such that $\mathbf{n}(\mathbf{r}) \cdot \mathbf{j} > 0$ for all $\mathbf{r} \in S$. Find the flux

$$\int_S [2y\mathbf{j} + z\mathbf{k}] dS.$$

