Chapter Nineteen

# Some Physics

## 19.1 Fluid Mechanics

Suppose $\mathbf{v}(x, y, z, t)$ is the velocity at $\mathbf{r} = (x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ of a fluid flowing smoothly through a region in space, and suppose $r(x, y, z, t)$ is the density at $\mathbf{r}$ at time $t$. If $S$ is an oriented surface, it is not hard to convince yourself that the flux integral

$$\iint_S r\mathbf{v} \cdot d\mathbf{r}$$

is the rate at which mass flows through the surface $S$. Now, if $S$ is a closed surface, then the mass in the region $B$ bounded by $S$ is, of course

$$\iiint_B r\,dV \; .$$

The rate at which this mass is changing is simply

$$\frac{\partial}{\partial t}\iiint_B r\,dV = \iiint_B \frac{\partial r}{\partial t}\,dV \; .$$

This is the same as the rate at which mass is flowing across $S$ into $B$: $-\iint_S r\mathbf{v} \cdot d\mathbf{r}$, where $S$ is given the outward pointing orientation. Thus,

$$\iiint_B \frac{\partial r}{\partial t}\,dV = -\iint_S r\mathbf{v} \cdot d\mathbf{r} \; .$$

We now apply Gauss's Theorem and get

$$\iiint_B \frac{\partial r}{\partial t}\,dV = -\iint_S r\mathbf{v} \cdot d\mathbf{r} = \iiint_B -\nabla \cdot (r\mathbf{v})\,dV .$$

Thus,

$$\iiint_B \left(\frac{\partial r}{\partial t} + \nabla \cdot (r\mathbf{v})\right) dV \; .$$

Meditate on this result. The region $B$ is *any* region, and so it must be true that the integrand itself is everywhere 0:

$$\frac{\partial r}{\partial t} + \nabla \cdot (r\mathbf{v}) = 0 .$$

This is one of the fundamental equations of fluid dynamics. It is called the ***equation of continuity***.

In case the fluid is incompressible, the continuity equation becomes quite simple. Incompressible means simply that the density $r$ is constant. Thus $\frac{\partial r}{\partial t} = 0$ and so we have

$$\frac{\partial r}{\partial t} + \nabla \cdot (r\mathbf{v}) = \nabla \cdot (r\mathbf{v}) = r\nabla \cdot \mathbf{v} = 0, \text{ or}$$

$$\nabla \cdot \mathbf{v} = 0 .$$

**Exercise**

**1.** Consider a one dimensional flow in which the velocity of the fluid is given by $\mathbf{v} = f(x)$, where $f(x) > 0$. Suppose further that the density $r$ of the fluid does not vary with time *t*. Show that

$$r(x) = \frac{k}{f(x)},$$

where *k* is a constant.

**19.2 Electrostatics**

Suppose there is a point charge *q* fixed at the point ***s***. Then the electric field $\mathbf{E}_q(\mathbf{r})$ due to *q* is given by

$$\mathbf{E}_q(\mathbf{r}) = kq\frac{\mathbf{r}-\mathbf{s}}{|\mathbf{r}-\mathbf{s}|^3} .$$

It is easy to verify, as we have done in a previous chapter, that this field, or function, is conservative, with a potential function

$$P_q(\mathbf{r}) = \frac{-kq}{|\mathbf{r}-\mathbf{s}|};$$

so that $\mathbf{E}_q = \nabla P_q$. Physicists do not like to be bothered with the minus sign in $P_q$, so they define the electric potential $V_q$ to be -$P_q$. Thus,

$$V_q(\mathbf{r}) = \frac{kq}{|\mathbf{r}-\mathbf{s}|},$$

and

$$\mathbf{E}_q(\mathbf{r}) = -\nabla V_q(\mathbf{r}) .$$

We have already seen that the flux out of a closed surface $S$ is

$$\iint_S \mathbf{E}_q \cdot d\mathbf{S} = \begin{cases} 0 & \text{if } S \text{ does not enclose the origin} \\ 4pkq & \text{if } S \text{ does enclose the origin} \end{cases}$$

Some meditation will convince you there is nothing special here about the origin; that is, if the point charge is at **s**, then

$$\iint_S \mathbf{E}_q \cdot d\mathbf{S} = \begin{cases} 0 & \text{if } S \text{ does not enclose } \mathbf{s} \\ 4pkq & \text{if } S \text{ does enclose } \mathbf{s} \end{cases}$$

Next, suppose there are a finite number of point charges $q_1$ at $\mathbf{s}_1, q_2$ at $\mathbf{s}_2, \ldots,$ and $q_n$ at $\mathbf{s}_n$. Suppose $\mathbf{E}_j$ is the electric intensity due to $q_j$. Then it should be clear that the electric field due to these charges is simply the sum

$$\mathbf{E}(\mathbf{r}) = \sum_{j=1}^{n} \mathbf{E}_j = k \sum_{j=1}^{n} q_j \frac{\mathbf{r}-\mathbf{s}_j}{|\mathbf{r}-\mathbf{s}_j|^3} .$$

Also,

$$V(\mathbf{r}) = k \sum_{j=1}^{n} \frac{q_j}{|\mathbf{r}-\mathbf{s}_j|}; \text{ and}$$

$$\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r}).$$

Finally,

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = 4pk \sum q_j$$

where the sum is over those charges enclosed by $S$.

Things become more exciting if instead of point charges, we have a charge distribution in space with charge density $r$ . To find the electric field $\mathbf{E}(\mathbf{r})$ produced by this distribution of charge in space, we need to integrate:

$$\mathbf{E}(\mathbf{r}) = \iiint_U k r(\mathbf{s}) \frac{(\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3} dV_s \,.$$

But this appears to be a serious breach of decorum. We are integrating over everything, and at $\mathbf{s}=\mathbf{r}$ we have the dreaded 0 in the denominator. Thus what we see above is an ***improper*** integral—that is, it is actually a limit of integrals. Specifically, we integrate not over everything but over everything outside a spherical solid region of radius *a* centered at **r**. We then look at the limit as $a \to 0$ of this integral. With the integral for the electric field, this limit exists, and so there is no problem with 0 on the bottom of the integrand. In the same way, we are safe in writing for the potential

$$V(\mathbf{r}) = k \iiint_U \frac{r(\mathbf{s})}{|\mathbf{r}-\mathbf{s}|} dV_s \,.$$

Everything works nicely so that we also have

$$\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r}) \,.$$

If *R* is a solid region bounded by a closed surface *S*, then we can also integrate to get

$$(*) \quad \iint_S \mathbf{E} \cdot d\mathbf{S} = 4pk \iiint_R r(\mathbf{s}) dV .$$

The divergence of **E** is the troublesome item in extending matters to distributed charge. If we simply try to calculate the divergence by $div \iiint_U \text{stuff}\, dV = \iiint_U div(\text{stuff}) dV$ , then things go wrong because the improper integral of the divergence does not exist. Gauss saves the day. Let *R* be any region and let *S* be the closed surface bounding *R*. Then

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = \iiint_R \nabla \cdot \mathbf{E} \, dV \,.$$

But from equation (*) we have

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = 4pk \iiint_R r(\mathbf{s}) dV = \iiint_R \nabla \cdot \mathbf{E} \, dV \,.$$

This gives us

$$\iiint_R 4pkr\,dV = \iiint_R \nabla \cdot \mathbf{E}\,dV \text{ , or}$$

$$\iiint_R (\nabla \cdot \mathbf{E} - 4pkr)\,dV \text{ .}$$

But $R$ is *any* region, and so it must be true that

$$\nabla \cdot \mathbf{E} = 4pkr$$

for all **r**.

Finally, remembering that $\mathbf{E} = -\nabla V$ , we get

$$\nabla \cdot \mathbf{E} = -\nabla \cdot (\nabla V) = 4pkr \text{ ;}$$

$$\nabla^2 V = -4pkr \text{ , or}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -4pkr \text{ .}$$

This is the celebrated ***Poisson's Equation***, a justly famous partial differential equation, the study of which is beyond the scope of this course.

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