

## Chapter Two

# Vectors-Algebra and Geometry

### 2.1 Vectors

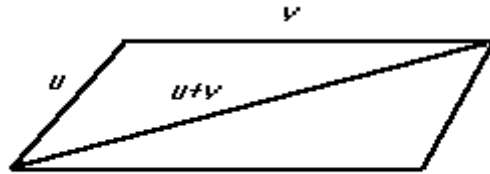
A *directed line segment* in space is a line segment together with a direction. Thus the directed line segment from the point  $P$  to the point  $Q$  is different from the directed line segment from  $Q$  to  $P$ . We frequently denote the direction of a segment by drawing an arrow head on it pointing in its direction and thus think of a directed segment as a spear. We say that two segments have the same direction if they are parallel and their directions are the same:



Here the segments  $L1$  and  $L2$  have the same direction. We define two directed segments  $L$  and  $M$  to be *equivalent* ( $L \sim M$ ) if they have the same direction and have the same length. An *equivalence class* containing a segment  $L$  is the set of all directed segments equivalent with  $L$ . Convince yourself every segment in an equivalence class is equivalent with every other segment in that class, and two different equivalence classes must be disjoint. These equivalence classes of directed line segments are called *vectors*. The members of a vector  $\mathbf{v}$  are called *representatives* of  $\mathbf{v}$ . Given a directed segment  $u$ , the vector which contains  $u$  is called the vector *determined* by  $u$ . *The length*, or *magnitude*, of a vector  $\mathbf{v}$  is defined to be the common length of the representatives of  $\mathbf{v}$ . It is generally designated by  $|\mathbf{v}|$ . The *angle* between two vectors  $\mathbf{u}$  and  $\mathbf{v}$  is simply the angle between the directions of representatives of  $\mathbf{u}$  and  $\mathbf{v}$ .

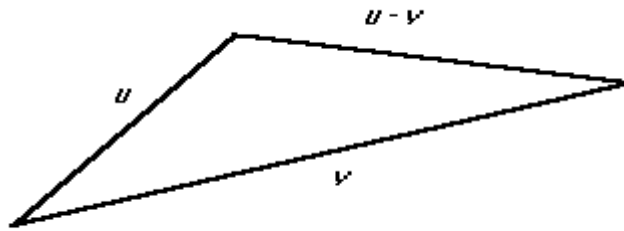
Vectors are just the right mathematical objects to describe certain concepts in physics. Velocity provides a ready example. Saying the car is traveling 50 miles/hour doesn't tell the whole story; you must specify in what direction the car is moving. Thus velocity is a vector-it has both magnitude and direction. Such physical concepts abound: force, displacement, acceleration, *etc.* The real numbers (or sometimes, the complex numbers) are frequently called *scalars* in order to distinguish them from vectors.

We now introduce an arithmetic, or algebra, of vectors. First, we define what we mean by the sum of two vectors  $\mathbf{u}$  and  $\mathbf{v}$ . Choose a spear  $u$  from  $\mathbf{u}$  and a spear  $v$  from  $\mathbf{v}$ . Place the tail of  $v$  at the nose of  $u$ . The vector which contains the directed segment from the tail of  $u$  to the nose of  $v$  is defined to be  $\mathbf{u} + \mathbf{v}$ , the sum of  $\mathbf{u}$  and  $\mathbf{v}$ . An easy consequence of elementary geometry is the fact that  $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$ . Look at the picture and convince yourself that the it does not matter which  $\mathbf{u}$  spear or  $\mathbf{v}$  spear you choose, and that  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$  :



Convince yourself also that addition is associative:  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ . Since it does not matter where the parentheses occur, it is traditional to omit them and write simply  $\mathbf{u} + \mathbf{v} + \mathbf{w}$ .

Subtraction is defined as the inverse operation of addition. Thus the difference  $\mathbf{u} - \mathbf{v}$  of two vectors is defined to be the vector you add to  $\mathbf{v}$  to get  $\mathbf{u}$ . In pictures, if  $u$  is a representative of  $\mathbf{u}$  and  $v$  is a representative of  $\mathbf{v}$ , and we put the tails of  $u$  and  $v$  together, the directed segment from the nose of  $v$  to the nose of  $u$  is a representative of  $\mathbf{u} - \mathbf{v}$ :



Now, what are we to make of  $\mathbf{u} - \mathbf{u}$ ? We define a special vector with 0 length, called the zero vector and denoted  $\mathbf{0}$ . We may think of  $\mathbf{0}$  as the collection of all degenerate line segments, or points. Note that the zero vector is special in that it has no direction (If you are going 0 miles/hour, the direction is not important!). To make our algebra of vectors nice, we make the zero vector behave as it should:

$$\mathbf{u} - \mathbf{u} = \mathbf{0} \text{ and } \mathbf{u} + \mathbf{0} = \mathbf{u}$$

for all vectors  $\mathbf{u}$ .

Next we define the product of a scalar  $r$  (*i.e.*, real number) with a vector  $\mathbf{u}$ . The product  $r\mathbf{u}$  is defined to be the vector with length  $|r||\mathbf{u}|$  and direction the same as the direction of  $\mathbf{u}$  if  $r > 0$ , and direction opposite the direction of  $\mathbf{u}$  if  $r < 0$ . Convince yourself that all the following nice properties of this multiplication hold:

$$(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u},$$

$$r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}.$$

$$0\mathbf{u} = \mathbf{0}, \text{ and}$$

$$\mathbf{u} + (-1)\mathbf{v} = \mathbf{u} - \mathbf{v}.$$

It is then perfectly safe to write  $-\mathbf{u}$  to stand for  $(-1)\mathbf{u}$ .

Our next move is to define a one-to-one correspondence between vectors and points in space (This will, of course, also establish a one-to-one correspondence between vectors and ordered triples of real numbers.). The correspondence is quite easy; simply take a representative of the vector  $\mathbf{u}$  and place its tail at the origin. The point at which is

found the nose of this representative is the point associated with  $\mathbf{u}$ . We handle the vector with no representatives by associating the origin with the zero vector. The fact that the point with coordinates  $(a, b, c)$  is associated with the vector  $\mathbf{u}$  in this manner is shorthandedly indicated by writing  $\mathbf{u} = (a, b, c)$ . Strictly speaking this equation makes no sense; an equivalence class of directed line segments cannot possibly be the same as a triple of real numbers, but this shorthand is usually clear and saves a lot of verbiage (The numbers  $a, b$ , and  $c$  are called the *coordinates*, or *components*, of  $\mathbf{u}$ ). Thus we frequently do not distinguish between points and vectors and indiscriminately speak of a *vector*  $(a,b,c)$  or of a point  $\mathbf{u}$ .

Suppose  $\mathbf{u} = (a, b, c)$  and  $\mathbf{v} = (x, y, z)$ . Unleash your vast knowledge of elementary geometry and convince yourself of the truth of the following statements:

$$|\mathbf{u}| = \sqrt{a^2 + b^2 + c^2} ,$$

$$\mathbf{u} + \mathbf{v} = (a + x, b + y, c + z),$$

$$\mathbf{u} - \mathbf{v} = (a - x, b - y, c - z), \text{ and}$$

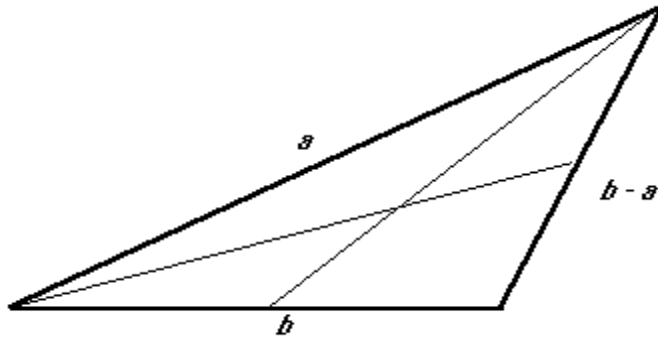
$$r\mathbf{u} = (ra, rb, rc).$$

Let  $\mathbf{i}$  be the vector corresponding to the point  $(1, 0, 0)$ ; let  $\mathbf{j}$  be the vector corresponding to  $(0, 1, 0)$ ; and let  $\mathbf{k}$  be the vector corresponding to  $(0, 0, 1)$ . Any vector  $\mathbf{u}$  can now be expressed as a linear combination of these special so-called *coordinate* vectors:

$$\mathbf{u} = (x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} .$$

### Example

Let's use our new-found knowledge of vectors to find where the medians of a triangle intersect. Look at the picture:



We shall find scalars  $s$  and  $t$  so that

$$\mathbf{a} + t\left(\frac{\mathbf{b}}{2} - \mathbf{a}\right) = s\left(\mathbf{a} + \frac{\mathbf{b} - \mathbf{a}}{2}\right).$$

Tidying this up gives us

$$\left(1 - t - \frac{s}{2}\right)\mathbf{a} = \left(\frac{s}{2} - \frac{t}{2}\right)\mathbf{b}.$$

This means that we must have

$$\frac{s}{2} - \frac{t}{2} = 0, \text{ and}$$

$$1 - t - \frac{s}{2} = 0.$$

Otherwise,  $\mathbf{a}$  and  $\mathbf{b}$  would be nonzero scalar multiples of one another, which would mean they have the same direction. It follows that

$$s = t = \frac{2}{3}.$$

This is, no doubt, the result you remember from Mrs. Turner's high school geometry class.

## Exercises

1. Find the vector such that if its tail is at the point  $(x_1, y_1, z_1)$  its nose will be at the point  $(x_2, y_2, z_2)$ .
2. Find the midpoint of the line segment joining the points  $(1, 5, 9)$  and  $(-3, 2, 3)$ .
3. What is the distance between the points  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$ ?
4. Describe the set of points  $L = \{t\mathbf{i} : - < t < \}$ .
5. Let  $\mathbf{u} = (2, 3, 8)$ . Describe the set of points  $L = \{t\mathbf{u} : - < t < \}$ .
6. Describe the set of points  $M = \{3\mathbf{k} + t\mathbf{i} : - < t < \}$ .
7. Let  $\mathbf{u} = (2, 3, 8)$  and  $\mathbf{v} = (1, 5, 7)$ . Describe the set of points  $M = \{\mathbf{v} + t\mathbf{u} : - < t < \}$ .
8. Describe the set  $P = \{t\mathbf{i} + s\mathbf{j} : - < t < , \text{ and } - < s < \}$ .
9. Describe the set  $P = \{5\mathbf{k} + t\mathbf{i} + s\mathbf{j} : - < t < , \text{ and } - < s < \}$ .
10. Let  $\mathbf{u} = (2, -4, 1)$  and  $\mathbf{v} = (1, 2, 3)$ . Describe the set  $P = \{t\mathbf{u} + s\mathbf{v} : - < t < , \text{ and } - < s < \}$ .
11. Let  $\mathbf{u} = (2, -4, 1)$ ,  $\mathbf{v} = (1, 2, 3)$ , and  $\mathbf{w} = (3, 6, 1)$ . Describe the set  $P = \{\mathbf{w} + t\mathbf{u} + s\mathbf{v} : - < t < , \text{ and } - < s < \}$ .
12. Describe the set  $C = \{\cos t \mathbf{i} + \sin t \mathbf{j} : 0 \leq t \leq 2\pi\}$ .

13. Describe the set  $E = \{4 \cos t \mathbf{i} + 3 \sin t \mathbf{j} : 0 \leq t \leq 2\pi\}$ .

14. Describe the set  $P = \{t \mathbf{i} + t^2 \mathbf{j} : -1 \leq t \leq 2\}$ .

15. Let  $T$  be the triangle with vertices  $(2, 5, 7)$ ,  $(-1, 2, 4)$ , and  $(4, -2, -6)$ . Find the point at which the medians intersect.

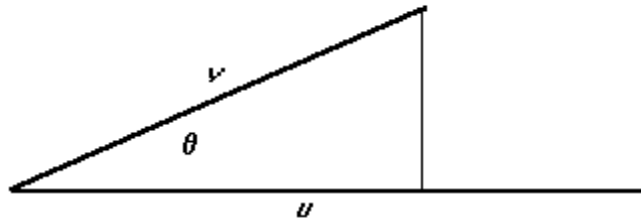
## 2.2 Scalar Product

You were perhaps puzzled when in grammar school you were first told that the work done by a force is the product of the force and the displacement since both force and displacement are, of course, vectors. We now introduce this product. It is a scalar and hence is called the *scalar product*. This scalar product  $\mathbf{u} \cdot \mathbf{v}$  is defined by

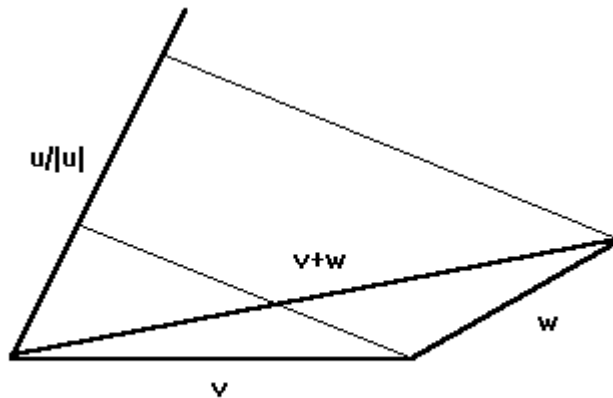
$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta,$$

where  $\theta$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$ . The scalar product is frequently also called the *dot product*. Observe that  $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$ , and that  $\mathbf{u} \cdot \mathbf{v} = 0$  if and only if  $\mathbf{u}$  and  $\mathbf{v}$  are perpendicular (or *orthogonal*), or one or the other of the two is the zero vector. We avoid having to use the latter weasel words by defining the zero vector to be perpendicular to every vector; then we can say  $\mathbf{u} \cdot \mathbf{v} = 0$  if and only if  $\mathbf{u}$  and  $\mathbf{v}$  are perpendicular.

Study the following picture to see that if  $|\mathbf{u}| = 1$ , then  $\mathbf{u} \cdot \mathbf{v}$  is the length of the projection of  $\mathbf{v}$  onto  $\mathbf{u}$ . (More precisely, the length of the projection of a representative of  $\mathbf{v}$  onto a representative of  $\mathbf{u}$ . Generally, where there is no danger of confusion, we omit mention of this, just as we speak of the length of vectors, the angle between vectors, *etc.*)



It is clear that  $(au) \cdot (bv) = (ab)u \cdot v$ . Study the following picture until you believe that  $u \cdot (v + w) = u \cdot v + u \cdot w$  for any three vectors  $u, v$ , and  $w$ .



Now let's get a recipe for the scalar product of  $u = (a, b, c)$  and  $v = (x, y, z)$ :

$$\begin{aligned} u \cdot v &= (ai + bj + ck) \cdot (xi + yj + zk) \\ &= axi \cdot i + ayi \cdot j + azj \cdot k + bxj \cdot i + byj \cdot j + bzj \cdot k + cxk \cdot i + cyk \cdot j + czk \cdot k \\ &= ax + by + cz, \end{aligned}$$

since  $i \cdot i = j \cdot j = k \cdot k = 1$  and  $i \cdot j = i \cdot k = j \cdot k = 0$ .

We thus see that it is remarkably simple to compute the scalar product of two vectors when we know their coordinates.

### Example



Again, let's see how vectors can make geometry easy by using them to find the angle between a diagonal of a cube and the diagonal of a face of the cube.

Suppose the cube has edge length  $s$ . Introduce a coordinate system so that the faces are parallel to the coordinate planes, one vertex is the origin and the vertex at the other end of the diagonal from the origin is  $(s, s, s)$ . The vector determined by this diagonal is thus  $\mathbf{d} = s\mathbf{i} + s\mathbf{j} + s\mathbf{k}$  and the vector determined by the diagonal of the face in the horizontal coordinate plane is  $\mathbf{f} = s\mathbf{i} + s\mathbf{j}$ . Thus

$$\mathbf{d} \cdot \mathbf{f} = |\mathbf{d}||\mathbf{f}|\cos \theta = s^2 + s^2,$$

where  $\theta$  is the angle we seek. This gives us

$$\cos \theta = \frac{2s^2}{|\mathbf{d}||\mathbf{f}|} = \frac{2s^2}{\sqrt{3s^2}\sqrt{2s^2}} = \frac{\sqrt{2}}{\sqrt{3}}.$$

Or,

$$\theta = \cos^{-1} \frac{\sqrt{2}}{\sqrt{3}}.$$

## Exercises

- 16.** Find the work done by the force  $\mathbf{F} = 6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$  in moving an object from the point  $(1, 4, -2)$  to the point  $(3, 2, 5)$ .
- 17.** Let  $L$  be the line passing through the origin and the point  $(2, 5)$ , and let  $M$  be the line passing through the points  $(3, -2)$  and  $(5, 3)$ . Find the smaller angle between  $L$  and  $M$ .
- 18.** Find an angle between the lines  $3x + 2y = 1$  and  $x - 2y = 3$ .

19. Suppose  $L$  is the line passing through  $(1, 2)$  having slope  $-2$ , and suppose  $M$  is the line tangent to the curve  $y = x^3$  at the point  $(1, 1)$ . Find the smaller angle between  $L$  and  $M$ .

20. Find an angle between the diagonal and an adjoining edge of a cube.

21. Suppose the lengths of the sides of a triangle are  $a$ ,  $b$ , and  $c$ ; and suppose  $\theta$  is the angle opposite the side having length  $c$ . Prove that

$$c^2 = a^2 + b^2 - 2ab \cos \theta .$$

(This is, of course, the celebrated *Law of Cosines*.)

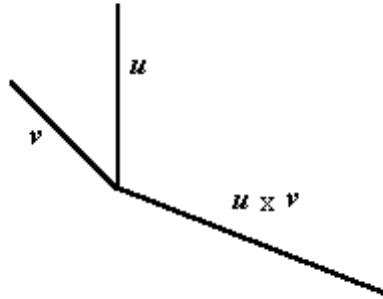
22. Let  $\mathbf{v} = (1, 2, 5)$ . What is the graph of the equation  $\mathbf{v} \cdot (x, y, z) = 0$ ?

### 2.3 Vector Product

Hark back to grammar school physics once again and recall what you were taught about the velocity of a point at a distance  $r$  from the axis of rotation; you were likely told that the velocity is  $r\omega$ , where  $\omega$  is the rate at which the turntable is rotating—the so-called angular velocity. We now know that these quantities are actually vectors— $\omega$  is the angular *velocity*, and  $\mathbf{r}$  is the *position* vector of the point in question. The grammar school quantities are the magnitudes of  $\omega$  (the angular *speed*) and of  $\mathbf{r}$ . The *velocity* of the point is the so-called *vector product* of these two vectors. The *vector product* of vectors  $\mathbf{u}$  and  $\mathbf{v}$  is defined by

$$\mathbf{u} \times \mathbf{v} = |\mathbf{u}||\mathbf{v}|\sin \theta \mathbf{n} ,$$

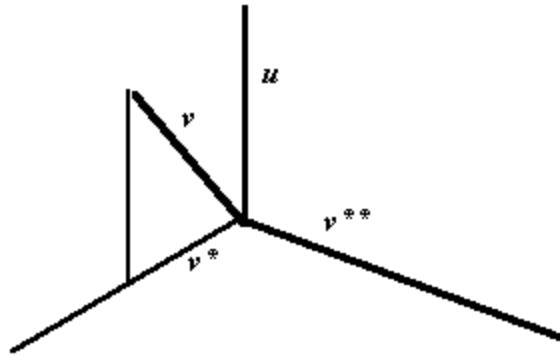
where  $\theta$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$  and  $\mathbf{n}$  is a vector of length 1 (such vectors are called *unit vectors*) which is orthogonal to both  $\mathbf{u}$  and  $\mathbf{v}$  and which points in the direction a right-hand threaded bolt would advance if  $\mathbf{u}$  were rotated into the direction of  $\mathbf{v}$ .



Note first that this is a somewhat more exciting product than you might be used to:

the *order* of the factors makes a difference. Thus  $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$ .

Now let's find a geometric construction of the vector product  $\mathbf{u} \times \mathbf{v}$ . Proceed as follows. Let  $P$  be a plane perpendicular to  $\mathbf{u}$ . Now project  $\mathbf{v}$  onto this plane, giving us a vector  $\mathbf{v}^*$  perpendicular to  $\mathbf{u}$  and having length  $|\mathbf{v}|\sin \theta$ . Now rotate this vector  $\mathbf{v}^*$  90 degrees around  $\mathbf{u}$  in the "positive direction." (By the positive direction of rotation about a vector  $\mathbf{a}$ , we mean the direction that would cause a right-hand threaded bolt to advance in the direction of  $\mathbf{a}$ .) This gives a vector  $\mathbf{v}^{**}$  having the same length as  $\mathbf{v}^*$  and having the direction of  $\mathbf{u} \times \mathbf{v}$ . Thus  $\mathbf{u} \times \mathbf{v} = |\mathbf{u}|\mathbf{v}^{**}$ :



Now, why did we go to all this trouble to construct  $\mathbf{u} \times \mathbf{v}$  in this fashion? Simple. It makes it much easier to see that for any three vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$ , we have

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} .$$

(Draw a picture!)

We shall see how to compute this vector product  $\mathbf{u} \times \mathbf{v}$  for

$$\mathbf{u} = (a, b, c) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \text{ and } \mathbf{v} = (x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} .$$

We have

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\ &= ax(\mathbf{i} \times \mathbf{i}) + ay(\mathbf{i} \times \mathbf{j}) + az(\mathbf{i} \times \mathbf{k}) + \\ &\quad bx(\mathbf{j} \times \mathbf{i}) + by(\mathbf{j} \times \mathbf{j}) + bz(\mathbf{j} \times \mathbf{k}) + \\ &\quad cx(\mathbf{k} \times \mathbf{i}) + cy(\mathbf{k} \times \mathbf{j}) + cz(\mathbf{k} \times \mathbf{k}) \end{aligned}$$

This looks like a terrible mess, until we note that

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0} ,$$

$$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}) = \mathbf{k} ,$$

$$\mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j}) = \mathbf{i} , \text{ and}$$

$$\mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k}) = \mathbf{j} .$$

Making these substitutions in the above equation for  $\mathbf{u} \times \mathbf{v}$  gives us

$$\mathbf{u} \times \mathbf{v} = (bz - cy)\mathbf{i} + (cx - az)\mathbf{j} + (ay - bx)\mathbf{k}.$$

This is not particularly hard to remember, but there is a nice memory device using determinants:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ x & y & z \end{vmatrix}.$$

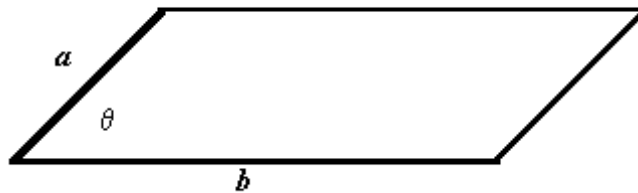
### Example

Let's find the velocity of a point on the surface of the Earth relative to a coordinate system whose origin is fixed at its center—we thus shall consider only motion due to the Earth's rotation, and neglect its motion about the sun, *etc.* For our point on the Earth, choose Room 254, Skiles Classroom Building at Georgia Tech. The latitude of the room is about 33.75 degrees (North, of course.), and it is about 3960 miles from the center of the Earth. As we said, the origin of our coordinate system is the center of the Earth. We choose the third axis to point through the North Pole; In other words, the coordinate vector  $\mathbf{k}$  points through the North Pole. The velocity of our room, is of course, not a constant, but changes as the Earth rotates. We find the velocity at the instant our room is in the coordinate plane determined by the vectors  $\mathbf{i}$  and  $\mathbf{k}$ .

The Earth makes one complete revolution every 24 hours, and so its angular velocity is  $= \frac{2\pi}{24}\mathbf{k} \approx 0.2618\mathbf{k}$  radians/hour. The position vector  $\mathbf{r}$  of our room is  $\mathbf{r} = 3960(\cos(33.75)\mathbf{i} + \sin(33.75)\mathbf{k}) \approx 3292.6\mathbf{i} + 2200.1\mathbf{k}$  miles. Our velocity is thus

$$\times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0.2618 \\ 32926 & 0 & 22001 \end{vmatrix} = 862\mathbf{j} \text{ miles/hour.}$$

Suppose we want to find the area of a parallelogram, the non-parallel sides of which are representatives of the vectors  $\mathbf{a}$  and  $\mathbf{b}$ :



The area  $A$  is clearly  $A = |\mathbf{a}||\mathbf{b}|\sin \theta = |\mathbf{a} \times \mathbf{b}|$ .

### Example

Find the area of the parallelogram with a vertex  $(1, 4, -2)$  and the vertices at the other ends of the sides adjoining this vertex are  $(4, 7, 8)$ , and  $(6, 10, 20)$ . This is easy. This is just as in the above picture with  $\mathbf{a} = (4 - 1)\mathbf{i} + (7 - 4)\mathbf{j} + (8 - (-2))\mathbf{k} = 3\mathbf{i} + 3\mathbf{j} + 10\mathbf{k}$  and  $\mathbf{b} = (6 - 1)\mathbf{i} + (10 - 4)\mathbf{j} + (20 - (-2))\mathbf{k} = 5\mathbf{i} + 6\mathbf{j} + 22\mathbf{k}$ . So we have

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 10 \\ 5 & 6 & 22 \end{vmatrix} = 6\mathbf{i} - 16\mathbf{j} + 3\mathbf{k},$$

and so,

$$\text{Area} = |\mathbf{a} \times \mathbf{b}| = \sqrt{6^2 + 16^2 + 3^2} = \sqrt{301}.$$

### Exercises

23. Find a vector perpendicular to the plane containing the points  $(1,4,6)$ ,  $(-1,2,-7)$ , and  $(-3,6,10)$ .

24. Are the points  $(0,4,7)$ ,  $(2, 6, 8)$ , and  $(5, 10, 20)$  collinear? Explain how you know?

25. Find the torque created by the force  $\mathbf{f} = 3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$  acting at the point  $\mathbf{a} = \mathbf{i} - 2\mathbf{j} - 7\mathbf{k}$ .

26. Find the area of the triangle whose vertices are  $(0,0,0)$ ,  $(1,2,3)$ , and  $(4,7,12)$ .

27. Find the volume of the parallelepiped

