

Chapter Three

Vector Functions

3.1 Relations and Functions

We begin with a review of the idea of a function. Suppose A and B are sets. The **Cartesian product** $A \times B$ of these sets is the collection of all ordered pairs (a,b) such that $a \in A$ and $b \in B$. A **relation** R is simply a subset of $A \times B$. The **domain** of R is the set $\text{dom } R = \{a \in A : (a,b) \in R\}$. In case $A = B$ and the domain of R is all of A , we call R a **relation on A** . A relation $R \subseteq A \times B$ such that $(a,b) \in R$ and $(a,c) \in R$ only if $b = c$ is called a **function**. In other words, if R is a function, and $a \in \text{dom } R$, there is exactly one ordered pair $(a,b) \in R$. The second “coordinate” b is thus uniquely determined by a . It is usually denoted $R(a)$. If $R \subseteq A \times B$ is a relation, the **inverse** of R is the relation $R^{-1} \subseteq B \times A$ defined by $R^{-1} = \{(b,a) : (a,b) \in R\}$.

Example

Let A be the set of all people who have ever lived and let $S \subseteq A \times A$ be the relation defined by $S = \{(a,b) : b \text{ is the mother of } a\}$. The S is a relation on A , and is, in fact, a function. The relation S^{-1} is not a function, and $\text{dom } S^{-1} \neq A$.

The fact that $f \subseteq A \times B$ is a function with $\text{dom } f = A$ is frequently indicated by writing $f: A \rightarrow B$, and we say f is a function **from** A to B . Very often a function f is defined by specifying the domain, and giving a recipe for finding $f(a)$. Thus we may define the function f from the interval $[0,1]$ to the real numbers by $f(x) = x^2$. This says that f is the collection of all ordered pairs (x,x^2) in which $x \in [0,1]$.

Exercises

1. Let A be the set of all Georgia Tech students, and let B be the set of real numbers. Define the relation $W \subset A \times B$ by $W = \{(a, b) : b \text{ is the weight (in pounds) of } a\}$. Is W a function? Is W^{-1} a function? Explain.
2. Let X be set of all states of the U. S., and let Y be the set of all U. S. municipalities. Define the relation $c \subset X \times Y$ by $c = \{(x, y) : y \text{ is the capital of } x\}$. Explain why c is a function, and find $c(\text{Nevada})$, $c(\text{Missouri})$, and $c(\text{Kentucky})$.
3. With X, Y as in Exercise 2, let b be the function $b = \{(x, y) : y \text{ is the largest city in } x\}$.
 - a) What is $b(\text{South Carolina})$?
 - b) What is $b(\text{California})$?
 - c) Let $f = c \circ b$, where c is the function defined in Exercise 2. Find $\text{dom } f$.
4. Suppose $f \subset X \times Y$ and $g \subset X \times Y$. If f is a function, is it necessarily true that $f \cap g$ is a function? Prove your answer.
5. Suppose $f \subset X \times Y$ and $g \subset X \times Y$. If f and g are both functions, is it necessarily true that $f \cup g$ is a function? Prove your answer.
6. Suppose $f : X \rightarrow Y$ is a function and the inverse f^{-1} is also a function.
 - a) What is $f^{-1}(f(x))$? Explain.
 - b) If $y \in \text{dom } f^{-1}$, what is $f(f^{-1}(y))$? Explain.

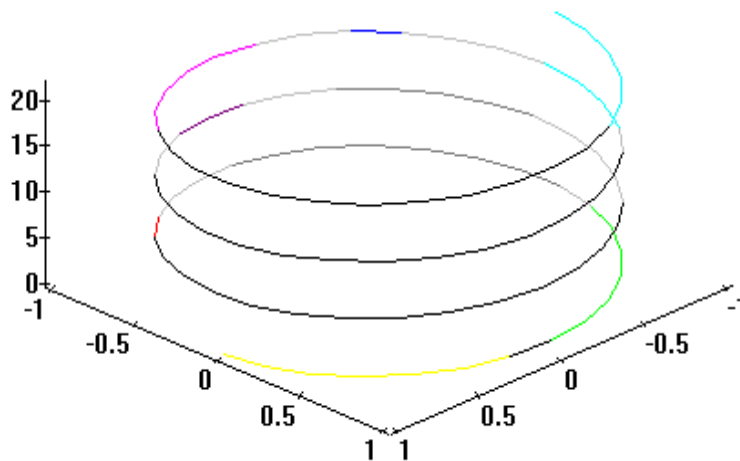
3.2 Vector Functions

Our interest now will be focused on functions $f : X \rightarrow Y$ in which Y is a set of vectors. These are called **vector functions**, or sometimes, **vector-valued** functions. Initially, we shall be solely interested in the special case in which X is a “nice” set of real numbers, such as an interval. As the drama unfolds, we shall see that such functions provide just the right tool for describing curves in space.

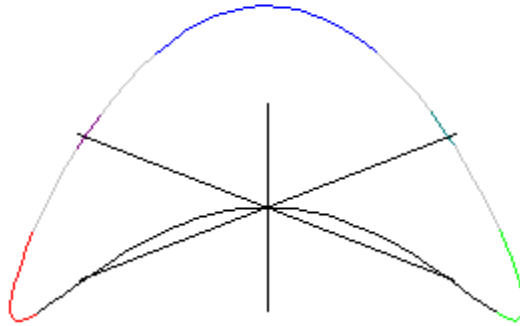
Let’s begin with a simple example. Let X be the entire real line and let the function f be defined by $f(t) = t\mathbf{i} + t^2\mathbf{j}$. It should be reasonably clear that if we place the tail of

$f(t)$ (actually, a representative of $f(t)$) at the origin, the nose will lie on the curve $y = x^2$. In fact, as t varies over the reals, the nose traces out this curve. The function f is called a **vector description** of the curve. Let's look at another example. This time, let $g(t) = \cos t \mathbf{i} + \sin t \mathbf{j}$ for $0 \leq t < 4\pi$. What is the curve described by this function? First, note that for all t , we have $|g(t)| = 1$. The nose of g thus always lies on the circle of radius one centered at the origin. It's not difficult to see that, in fact, as t varies from 0 to 2π , the nose moves around the circle once, and as t varies on from 2π to 4π , the nose traces out the circle again.

The real usefulness of vector descriptions is most evident when we consider curves in space. Let $f(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$, for all $t \geq 0$. Now, what curve is followed by the nose of $f(t)$? Notice first that if we look down on this curve from someplace up the positive third axis (In other words, \mathbf{k} is pointing directly at us.), we see the circle described by $\cos t \mathbf{i} + \sin t \mathbf{j}$. As t increases, we run around this circle and the third component of our position increases linearly. Convince yourself now that this curve looks like this:



This curve is called a **helix**, or more precisely, a **right circular helix**. The picture was drawn by *Maple*. Let's draw another. How about the curve described by the vector function $g(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + \sin(2t) \mathbf{k}$? This one is just a bit more exciting. Here's a computer drawn picture:



(This time we put the axes where they are “supposed to be.”)

Observe that in giving a vector description, we are in effect specifying the three coordinates of points on the curves as ordinary real valued functions defined on a subset of the reals. Assuming the axes are labeled x , y , and z , the curve described by the vector function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

is equivalently described by the equations

$$x = f(t)$$

$$y = g(t)$$

$$z = h(t)$$

These are called *parametric equations* of the curve (The variable t is called the *parameter*.).

Exercises

7. Sketch or otherwise describe the curve given by $\mathbf{f}(t) = t\mathbf{i} + t^3\mathbf{k}$ for $-1 \leq t \leq 3$.

8. Sketch or otherwise describe the curve given by $\mathbf{f}(t) = (2t - 3)\mathbf{i} + (3t + 1)\mathbf{j}$.

[Hint: Find an equation in x and y , the graph of which is the given curve.]

9. Sketch or otherwise describe the curve given by $\mathbf{c}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + 7\mathbf{k}$.
10. Sketch or otherwise describe the curve given by $\mathbf{c}(t) = \cos(t^2) \mathbf{i} + \sin(t^2) \mathbf{j} + 7\mathbf{k}$.
11. Find an equation in x and y , the graph of which is the curve $\mathbf{g}(t) = 3\cos t \mathbf{i} + 4\sin t \mathbf{j}$.
12. a) Find a vector equation for the graph of $y = x^3 + 2x^2 + x + 5$.
b) Find a vector equation for the graph of $x = y^3 + 2y^2 + y + 5$.
13. Find a vector equation for the graph of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.
14. a) Sketch or otherwise describe the curve given by the function $\mathbf{r}(t) = \mathbf{a} + t\mathbf{b}$, where $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$.
b) Express $\mathbf{r}(t)$ in the form $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$.
15. Describe the curve given by $\mathbf{L}(t) = (3t + 1)\mathbf{i} + (1 - t)\mathbf{j} + 2t\mathbf{k}$.
16. Find a vector function for the straight line passing through the point $(1, 4, -2)$ in the direction of the vector $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.
17. a) Find a vector function for the straight line passing through the points $(1, 2, 4)$ and $(3, 1, 5)$.
b) Find a vector function for the line segment joining the points $(1, 2, 4)$ and $(3, 1, 5)$.
18. Let L be the line through the points $(1, 5, -2)$ and $(2, 2, 4)$; and let M be the line through the points $(2, 4, 6)$ and $(-3, 1, -2)$. Find a vector description of the line which passes through the point $(1, 1, 2)$ and is perpendicular to both L and M .

3.3 Limits and Continuity

Recall from grammar school what we mean when we say the limit at t_0 of a real-valued, or scalar, function f is L . The definition for vector functions is essentially the same. Specifically, suppose \mathbf{f} is a vector valued function, t_0 is a real number, and \mathbf{L} is a vector such that for every real number $\epsilon > 0$, there is a $\delta > 0$ such that $|\mathbf{f}(t) - \mathbf{L}| < \epsilon$ whenever $0 < |t - t_0| < \delta$ and t is in the domain of \mathbf{f} . This is traditionally written

$$\lim_{t \rightarrow t_0} \mathbf{f}(t) = \mathbf{L}.$$

The vector \mathbf{L} is called a *limit of \mathbf{f} at a* .

Suppose $g(t)$ is a scalar function for which $\lim_{t \rightarrow t_0} g(t) = a$, and \mathbf{f} is a vector function for which $\lim_{t \rightarrow t_0} \mathbf{f}(t) = \mathbf{L}$. It is but a modest exercise to show that

$$\lim_{t \rightarrow t_0} (g(t)\mathbf{f}(t)) = a\mathbf{L}.$$

To see this, we use the “behold!” method. Let $\epsilon > 0$ be given. Choose $\epsilon_1, \epsilon_2, \epsilon_3$, and ϵ_4 so that

$$|\mathbf{f}(t) - \mathbf{L}| < \frac{\epsilon_1}{3(1+|a|)} \text{ for } 0 < |t - t_0| < \delta_1;$$

$$|\mathbf{f}(t) - \mathbf{L}| < \sqrt{\frac{\epsilon_2}{3}} \text{ for } 0 < |t - t_0| < \delta_2;$$

$$|g(t) - a| < \frac{\epsilon_3}{3(1+|\mathbf{L}|)} \text{ for } 0 < |t - t_0| < \delta_3; \text{ and}$$

$$|g(t) - a| < \sqrt{\frac{\epsilon_4}{3}} \text{ for } 0 < |t - t_0| < \delta_4.$$

Now let $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$ suppose t is such that $0 < |t - t_0| < \delta$. Then

$$\begin{aligned} |g(t)\mathbf{f}(t) - a\mathbf{L}| &= |a(\mathbf{f}(t) - \mathbf{L}) + \mathbf{L}(g(t) - a) + (g(t) - a)(\mathbf{f}(t) - \mathbf{L})| \\ &\leq |a(\mathbf{f}(t) - \mathbf{L})| + |\mathbf{L}(g(t) - a)| + |(g(t) - a)(\mathbf{f}(t) - \mathbf{L})| \\ &< \frac{|a|\epsilon_1}{3(1+|a|)} + \frac{|\mathbf{L}|\epsilon_3}{3(1+|\mathbf{L}|)} + \sqrt{\frac{\epsilon_2}{3}}\sqrt{\frac{\epsilon_4}{3}} < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

Or, in other words,

$$\lim_{t \rightarrow t_0} (t) \mathbf{f}(t) = a\mathbf{L} ,$$

which is what we set out to show.

Now suppose $\mathbf{f}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ and $\mathbf{L} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Then we see that $\lim_{t \rightarrow t_0} \mathbf{f}(t) = \mathbf{L}$ if and only if

$$\begin{aligned} \lim_{t \rightarrow t_0} x(t) &= a, \\ \lim_{t \rightarrow t_0} y(t) &= b, \text{ and} \\ \lim_{t \rightarrow t_0} z(t) &= c. \end{aligned}$$

It is now easy to show that all the usual nice properties of limits are valid for vector functions:

$$\begin{aligned} \lim_{t \rightarrow t_0} (\mathbf{f}(t) + \mathbf{g}(t)) &= \lim_{t \rightarrow t_0} \mathbf{f}(t) + \lim_{t \rightarrow t_0} \mathbf{g}(t). \\ \lim_{t \rightarrow t_0} (\mathbf{f}(t) \cdot \mathbf{g}(t)) &= (\lim_{t \rightarrow t_0} \mathbf{f}(t)) \cdot (\lim_{t \rightarrow t_0} \mathbf{g}(t)). \\ \lim_{t \rightarrow t_0} (\mathbf{f}(t) \times \mathbf{g}(t)) &= (\lim_{t \rightarrow t_0} \mathbf{f}(t)) \times (\lim_{t \rightarrow t_0} \mathbf{g}(t)). \end{aligned}$$

We are now ready to say what we mean by a vector function's being continuous at a point of its domain. Suppose t_0 is in the domain of the vector function \mathbf{f} . Then we say \mathbf{f} is *continuous at t_0* if it is true that $\lim_{t \rightarrow t_0} \mathbf{f}(t) = \mathbf{f}(t_0)$. It is easy to see that if

$$\mathbf{f}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} ,$$

then \mathbf{f} is continuous at t_0 if and only if each of the everyday scalar functions $x(t)$, $y(t)$, and $z(t)$ is continuous at t_0 . This shows there is nothing particularly mysterious or exotic about continuity of vector functions.

If \mathbf{f} is continuous at each point of its domain, then we say simply that \mathbf{f} is *continuous*,

Exercises

19. Is it possible for a function f to have more than one limit at $t = t_0$? Prove your answer.
20. Suppose m is a continuous real-valued function and f is a continuous vector-valued function. Is the vector function h defined by $h(t) = m(t)f(t)$ also continuous? Explain.
21. Let f and g be continuous at $t = t_0$. Is the function h defined by $h(t) = f(t) \times g(t)$ continuous? Explain. How about the function $r(t) = f(t) \cdot g(t)$?
22. Let $r(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{t}\mathbf{k}$. Is r a continuous function? Explain.
23. Suppose r is a continuous function. Explain how you know that the length function $n(t) = |r(t)|$ is continuous.