

Chapter Four

Derivatives

4.1 Derivatives

Suppose \mathbf{f} is a vector function and t_0 is a point in the interior of the domain of \mathbf{f} (t_0 in the interior of a set S of real numbers means there is an interval centered at t_0 that is a subset of S). The derivative is defined just as it is for a plain old everyday real valued function, except, of course, the derivative is a vector. Specifically, we say that \mathbf{f} is *differentiable at t_0* if there is a vector \mathbf{v} such that

$$\lim_{h \rightarrow 0} \frac{1}{h} [\mathbf{f}(t_0 + h) - \mathbf{f}(t_0)] = \mathbf{v}.$$

The vector \mathbf{v} is called the *derivative of \mathbf{f} at t_0* .

Now, how would we find such a thing? Suppose $\mathbf{f}(t) = a(t)\mathbf{i} + b(t)\mathbf{j} + c(t)\mathbf{k}$.

Then

$$\frac{1}{h} [\mathbf{f}(t_0 + h) - \mathbf{f}(t_0)] = \frac{a(t_0 + h) - a(t_0)}{h} \mathbf{i} + \frac{b(t_0 + h) - b(t_0)}{h} \mathbf{j} + \frac{c(t_0 + h) - c(t_0)}{h} \mathbf{k}.$$

It should now be clear that the vector function \mathbf{f} is differentiable at t_0 if and only if each of the coordinate functions $a(t)$, $b(t)$, and $c(t)$ is. Moreover, the vector derivative \mathbf{v} is $\mathbf{v} = a'(t)\mathbf{i} + b'(t)\mathbf{j} + c'(t)\mathbf{k}$.

Now we “know” what the derivative of a vector function is, and we know how to compute it, but what is it, really? Let’s see. Let $\mathbf{f}(t) = t\mathbf{i} + t^3\mathbf{j}$. This is, of course, a vector function which describes the graph of the function $y = x^3$. Let’s look at the

derivative of f at t_0 : $\mathbf{v} = \mathbf{i} + 3t_0^2\mathbf{j}$. Convince yourself that the direction of the vector \mathbf{v} is the direction tangent to the graph of $y = x^3$ at the point (t_0, t_0^3) . It is not so clear what we should define to be the tangent to a curve other than a plane curve. Again, vectors come to our rescue. If \mathbf{f} is a vector description of a space curve, the direction of the derivative $\mathbf{f}'(t)$ vector is the *tangent* direction at the point $\mathbf{f}(t)$ -the derivative $\mathbf{f}'(t)$ is said to be tangent to the curve at $\mathbf{f}(t)$.

If $\mathbf{f}(t)$ specifies the position of a particle at time t , then, of course, the derivative is the *velocity* of the particle, and its length $|\mathbf{f}'(t)|$ is the *speed*. Thus the distance the particle travels from time $t = a$ to time $t = b$ is given by the integral of the speed:

$$d = \int_a^b |\mathbf{f}'(t)| dt.$$

If the particle behaves nicely, this distance is precisely the length of the arc of the curve from $\mathbf{f}(a)$ to $\mathbf{f}(b)$. It should be clear what we mean by “behaves nicely”. . For the distance traveled by the particle to be the same as the length of its path, there must be no “backtracking”, or reversing direction. This means we must not allow the velocity to be zero for any t between a and b .

Example

Consider the function $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$. Then the derivative, or velocity, is $\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$. This vector is indeed tangent to the curve described by \mathbf{r} (which we already know to be a circle of radius 1 centered at the origin.) at $\mathbf{r}(t)$. Note that the scalar product $\mathbf{r}(t) \cdot \mathbf{r}'(t) = -\sin t \cos t + \sin t \cos t = 0$, and so the tangent vector and the vector from the center of the circle to the point on the circle are perpendicular-a well-known fact you learned from Mrs. Turner in 4th grade. Note that the derivative is never

zero—there is no value of t for which both $\cos t$ and $\sin t$ vanish. The length of a piece of the curve can thus be found by integrating the speed:

$$p = \int_0^2 |\mathbf{r}'(t)| dt = \int_0^2 \sqrt{\sin^2 t + \cos^2 t} dt = \int_0^2 1 dt = 2 .$$

No surprise here.

Exercises

- Find a vector tangent to the curve $\mathbf{f}(t) = t^2\mathbf{i} + t^3\mathbf{j} + (1-t)\mathbf{k}$ at the point $(1, 1, 0)$.
 - Find a vector equation for the line tangent to this same curve at the point $(1, 1, 0)$.
- The position of a particle is given by $\mathbf{r}(t) = \cos(t^3)\mathbf{i} + \sin(t^3)\mathbf{j}$.

 - Find the velocity of the particle.
 - Find the speed of the particle.
 - Describe the path of the particle, and find its length.
- Let L be the line tangent to the curve $\mathbf{g}(t) = 10\cos t\mathbf{i} + 10\sin t\mathbf{j} + 16t\mathbf{k}$ at the point $(\frac{10}{\sqrt{2}}, \frac{10}{\sqrt{2}}, 4)$. Find the point at which L intersects the \mathbf{i} - \mathbf{j} plane.
- Let L be the straight line passing through the point $(5, 0, 3)$ in the direction of the vector $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, and let M be the straight line passing through the point $(0, 0, 6)$ in the direction of $\mathbf{b} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

 - Are L and M parallel? Explain.
 - Do L and M intersect? Explain.

5. Let L be the straight line passing through the point $(1, 1, 3)$ in the direction of the vector $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, and let M be the straight line passing through the point $(0, 1, 5)$ in the direction of $\mathbf{b} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Find the distance between L and M .
6. Find the length of the arc of the curve $\mathbf{R}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + 4t\mathbf{k}$ between the points $(3, 0, 0)$ and $(3, 0, 16)$.
7. Find an integral the value of which is the length of the curve $y = x^2$ between the points $(-1, 1)$ and $(1, 1)$.

4.2 Geometry of Space Curves-Curvature

Let $\mathbf{R}(t)$ be a vector description of a curve. Then the distance $s(t)$ along the curve from the point $\mathbf{R}(t_0)$ to the point $\mathbf{R}(t)$ is, as we have seen, simply

$$s(t) = \int_{t_0}^t |\mathbf{R}'(\tau)| d\tau ;$$

assuming, of course, that $\mathbf{R}'(t) \neq 0$. The speed is

$$\frac{ds}{dt} = |\mathbf{R}'(t)|.$$

Now then the vector

$$\mathbf{T} = \frac{\mathbf{R}'(t)}{|\mathbf{R}'(t)|} = \frac{\mathbf{R}'(t)}{ds/dt} = \mathbf{R}'(t) \frac{dt}{ds} = \frac{d\mathbf{R}}{ds}$$

is tangent to \mathbf{R} and has length one. It is called the *unit tangent vector*.

Consider next the derivative

$$\frac{d}{ds} \mathbf{T} \cdot \mathbf{T} = \mathbf{T} \cdot \frac{d\mathbf{T}}{ds} + \frac{d\mathbf{T}}{ds} \cdot \mathbf{T} = 2\mathbf{T} \cdot \frac{d\mathbf{T}}{ds}.$$

But we know that $\mathbf{T} \cdot \mathbf{T} = |\mathbf{T}|^2 = 1$. Thus $\mathbf{T} \cdot \frac{d\mathbf{T}}{ds} = 0$, which means that the vector $\frac{d\mathbf{T}}{ds}$ is perpendicular, or orthogonal, or normal, to the tangent vector \mathbf{T} . The length of this vector is called the *curvature* and is usually denoted by the letter κ . Thus

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|.$$

The unit vector

$$\mathbf{N} = \frac{1}{\kappa} \frac{d\mathbf{T}}{ds}$$

is called the *principal unit normal vector*, and its direction is sometimes called the *principal normal direction*.

Example

Consider the circle of radius a and center at the origin: $\mathbf{R}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j}$.

Then $\mathbf{R}'(t) = -a \sin t \mathbf{i} + a \cos t \mathbf{j}$, and $\frac{ds}{dt} = |\mathbf{R}'(t)| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} = \sqrt{a^2} = |a| = a$.

Thus

$$\mathbf{T} = \frac{1}{a} \mathbf{R}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j}.$$

Let's not stop now.

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt} \frac{dt}{ds} = \frac{1}{a} \frac{d\mathbf{T}}{dt} = \frac{1}{a} (-\cos t \mathbf{i} - \sin t \mathbf{j}).$$

Thus $\left| \frac{dT}{ds} \right| = \frac{1}{a}$, and $N = -(\cos t \mathbf{i} + \sin t \mathbf{j})$. So the curvature is the reciprocal of the radius and the principal normal vector points back toward the center of the circle.

Another Example

This time let $\mathbf{R} = (t+1)\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}$. First, $\mathbf{R}'(t) = \mathbf{i} + 2\mathbf{j} + 2t\mathbf{k}$, and so $\frac{ds}{dt} = |\mathbf{R}'(t)| = \sqrt{5+4t^2}$. The unit tangent is then

$$\mathbf{T} = \frac{1}{\sqrt{5+4t^2}} (\mathbf{i} + 2\mathbf{j} + 2t\mathbf{k}).$$

It's a bit of a chore now to find the curvature and the principal normal, so let's use a computer algebra system; *viz.*, *Maple*:

First, let's enter the unit tangent vector \mathbf{T} :

$$\mathbf{T} := t \rightarrow \frac{[1, 2, 2t]}{\text{sqrt}(5+4t^2)}$$

See if we got it right:

$\mathbf{T}(t);$

$$\frac{[1, 2, 2t]}{\sqrt{5+4t^2}}$$

Fine. Now differentiate:

$$\mathbf{A} := t \rightarrow \frac{\text{simplify}(\text{diff}(\mathbf{T}(t), t))}{\text{sqrt}(5+4t^2)}$$

$\mathbf{A}(t);$

$$\frac{5[0, 0, 2] + 4[0, 0, 2]t^2 - 4[1, 2, 2t]t}{(5+4t^2)^2}$$

We need to tidy this up:

$$B := t \rightarrow \frac{\text{evalm}(\text{numer}(A(t)))}{\text{denom}(A(t))}$$

B(t);

$$\frac{[-4t \ -8t \ 10]}{(5+4t^2)^2}$$

This vector is, of course, the normal $\frac{d\mathbf{T}}{ds}$. We continue and find the curvature and the principal normal \mathbf{N} .

kappa:=t->simplify(sqrt(dotprod(B(t),B(t))));

kappa(t);

$$2 \frac{\sqrt{5}}{(5+4t^2)^{3/2}}$$

$$N := t \rightarrow \frac{B(t)}{\kappa(t)}$$

N(t);

$$\frac{1}{10} \frac{[-4t \ -8t \ 10]\sqrt{5}}{\sqrt{5+4t^2}}$$

So there we have at last the speed $\frac{ds}{dt}$, the unit tangent \mathbf{T} , the curvature κ , and the principal normal \mathbf{N} .

Exercises

8. Find a line tangent to the curve $\mathbf{R}(t) = (t^2 + t)\mathbf{i} + (t + 1)\mathbf{j} - (t^3 + 5)\mathbf{k}$ and passing through the point (5, -2, 15), or show there is no such line.

9. Find the unit tangent T , the principal normal N , and the curvature κ , for the curves:

a) $\mathbf{R}(t) = 5\cos(t)\mathbf{i} + 5\sin(t)\mathbf{j} + 2t\mathbf{k}$

b) $\mathbf{R}(t) = (2t + 3)\mathbf{i} + (5 - t^2)\mathbf{j}$

c) $\mathbf{R}(t) = e^t \cos t \mathbf{i} + e^t \sin t \mathbf{j} + 6t \mathbf{k}$

10. Find the curvature of the curve $y = f(x)$ at $(x_0, f(x_0))$.

11. Find the curvature of $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j}$. At what point on the curve is the curvature the largest? smallest?

12. Find the curvature of $\mathbf{R}(t) = t\mathbf{i} + t^3\mathbf{j}$. At what point on the curve is the curvature the largest? smallest?

4.3 Geometry of Space Curves-Torsion

Let $\mathbf{R}(t)$ be a vector description of a curve. If T is the unit tangent and N is the principal unit normal, the unit vector $\mathbf{B} = T \times N$ is called the *binormal*. Note that the binormal is orthogonal to both T and N . Let's see about its derivative $\frac{d\mathbf{B}}{ds}$ with respect

to arclength s . First, note that $\mathbf{B} \cdot \mathbf{B} = |\mathbf{B}|^2 = 1$, and so $\mathbf{B} \cdot \frac{d\mathbf{B}}{ds} = 0$, which means that

being orthogonal to \mathbf{B} , the derivative $\frac{d\mathbf{B}}{ds}$ is in the plane of T and N . Next, note that \mathbf{B} is

perpendicular to the tangent vector T , and so $\mathbf{B} \cdot T = 0$. Thus $\frac{d\mathbf{B}}{ds} \cdot T = 0$. So what have

we here? The vector $\frac{d\mathbf{B}}{ds}$ is perpendicular to both \mathbf{B} and \mathbf{T} , and so must have the direction of \mathbf{N} (or, of course, $-\mathbf{N}$). This means

$$\frac{d\mathbf{B}}{ds} = -\mathbf{N}.$$

The scalar τ is called the *torsion*.

Example

Let's find the torsion of the helix $\mathbf{R}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j} + b t \mathbf{k}$. Here we go!

$\mathbf{R}'(t) = -a \sin t \mathbf{i} + a \cos t \mathbf{j} + b \mathbf{k}$. Thus $\frac{ds}{dt} = |\mathbf{R}'(t)| = \sqrt{a^2 + b^2}$, and we have

$$\mathbf{T} = \frac{1}{\sqrt{a^2 + b^2}} (-a \sin t \mathbf{i} + a \cos t \mathbf{j} + b \mathbf{k}).$$

Now then

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt} \frac{dt}{ds} = \frac{-a}{(a^2 + b^2)} (\cos t \mathbf{i} + \sin t \mathbf{j}).$$

Therefore,

$$= \frac{a}{(a^2 + b^2)} \text{ and } \mathbf{N} = -(\cos t \mathbf{i} + \sin t \mathbf{j}).$$

Let's don't stop now:

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{a^2 + b^2}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin t & a \cos t & b \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{1}{\sqrt{a^2 + b^2}} (b \sin t \mathbf{i} - b \cos t \mathbf{j} + a \mathbf{k});$$

and

$$\frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}}{dt} \frac{dt}{ds} = \frac{b}{(a^2 + b^2)} (\cos t \mathbf{i} + \sin t \mathbf{j}) = \frac{-b}{(a^2 + b^2)} \mathbf{N}.$$

The torsion, at last:

$$= \frac{b}{a^2 + b^2}.$$

Suppose the curve $\mathbf{R}(t)$ is such that the torsion is zero for all values of t . In other words, $\frac{d\mathbf{B}}{ds} = 0$. Look at

$$\frac{d}{ds} [(\mathbf{R}(t) - \mathbf{R}(t_0)) \cdot \mathbf{B}] = \frac{d\mathbf{R}}{ds} \cdot \mathbf{B} + (\mathbf{R}(t) - \mathbf{R}(t_0)) \cdot \frac{d\mathbf{B}}{ds} = 0.$$

Thus the scalar product $(\mathbf{R}(t) - \mathbf{R}(t_0)) \cdot \mathbf{B}$ is constant. It is 0 at t_0 , and hence it is 0 for all values of t . This means that $\mathbf{R}(t) - \mathbf{R}(t_0)$ and \mathbf{B} are perpendicular for all t , and so $\mathbf{R}(t) - \mathbf{R}(t_0)$ lies in a plane perpendicular to \mathbf{B} . In other words, the curve described by $\mathbf{R}(t)$ is a plane curve.

Exercises

13. Find the binormal and torsion for the curve $\mathbf{R}(t) = 4 \cos t \mathbf{i} + 3 \sin t \mathbf{k}$.

14. Find the binormal and torsion for the curve $\mathbf{R}(t) = \frac{\sin t}{\sqrt{2}} \mathbf{i} + \cos t \mathbf{j} + \frac{\sin t}{\sqrt{2}} \mathbf{k}$.

15. Find the curvature and torsion for $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$.

16. Show that the curve $\mathbf{R}(t) = t\mathbf{i} + \frac{1+t}{t}\mathbf{j} + \frac{1-t^2}{t}\mathbf{k}$ lies in a plane.

17. What is the vector $\mathbf{B} \times \mathbf{T}$? How about $\mathbf{N} \times \mathbf{T}$?

4.4 Motion

Suppose t is time and $\mathbf{R}(t)$ is the position vector of a body. Then the curve described by $\mathbf{R}(t)$ is the path, or trajectory, of the body, $\mathbf{v}(t) = \frac{d\mathbf{R}}{dt}$ is the velocity, and

$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt}$ is the acceleration. We know that $\mathbf{v}(t) = \frac{ds}{dt}\mathbf{T}$, and so the direction of the

velocity is the unit tangent \mathbf{T} . Let's see about the direction of the acceleration:

$$\begin{aligned} \mathbf{a}(t) &= \frac{d\mathbf{v}}{dt} = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} \\ &= \frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2 \mathbf{N} \end{aligned}$$

since $\frac{d\mathbf{T}}{dt} = \frac{ds}{dt}\mathbf{N}$. This tells us that the acceleration is always in the plane of the

vectors \mathbf{T} and \mathbf{N} . The derivative of the speed $\frac{d^2s}{dt^2}$ is the *tangential component* of the

acceleration, and $\left(\frac{ds}{dt}\right)^2$ is the *normal component* of the acceleration.

Example

Suppose a person who weighs 160 pounds moves around a circle having radius 20 feet at a constant speed of 60 miles/hour. What is the magnitude of the force on this person at any time?

First, we know the force \mathbf{f} is the mass times the acceleration: $\mathbf{f}(t) = m\mathbf{a}(t)$. Thus

$$\mathbf{f} = m \frac{d^2s}{dt^2} \mathbf{T} + m \frac{ds}{dt}^2 \mathbf{N}$$

also have The speed is a constant 60 miles/hour, or 88 feet/second; in other words,

$$\frac{ds}{dt} = 88 \text{ and } \frac{d^2s}{dt^2} = 0. \text{ Hence,}$$

$$|\mathbf{f}| = |m \frac{ds}{dt}^2 \mathbf{N}| = m \frac{ds}{dt}^2.$$

The mass $m = \frac{160}{32} = 5$ slugs, and the curvature $\kappa = \frac{1}{20}$. The magnitude of the force is

$$\text{thus } |\mathbf{f}| = \frac{5 \cdot 88^2}{20} = 1936 \text{ pounds.}$$

Exercises

18. The position of an object at time t is given by $\mathbf{r}(t) = t\mathbf{i} + (t^3 - 2)\mathbf{j} + 2t\mathbf{k}$. Find the velocity, the speed, and the tangential and normal components of the acceleration.

19. A force $\mathbf{f}(t) = t^2\mathbf{i} + (t - 1)\mathbf{j} + \mathbf{k}$ newtons is applied to an object of mass 2 kilograms.

At time $t = 0$, the object is at the origin. Find its position at time t .

20. A projectile of weight w is fired from the origin with an initial speed v_0 in the direction of the vector $\cos \theta \mathbf{i} + \sin \theta \mathbf{j}$, and the only force acting on the projectile is $\mathbf{f} = -w\mathbf{j}$.

a) Find a vector description of the trajectory of the projectile.

b) Find an equation the graph of which is the trajectory.

21. A 16 lb. bowling ball is rolled along a track with a circular vertical loop of radius a feet. What must the speed of the ball be in order for it not to fall from the track? What must the speed of an 8 lb. ball be in order for it not to fall?