

Chapter Eight

$$f: \mathbf{R}^n \rightarrow \mathbf{R}$$

8.1 Introduction

We shall now turn our attention to the very important special case of functions that are real, or scalar, valued. These are sometimes called *scalar fields*. In the very, but important, special subcase in which the dimension of the domain space is 2, we can actually look at the graph of a function. Specifically, suppose $f: \mathbf{R}^2 \rightarrow \mathbf{R}$. The collection $S = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : f(x_1, x_2) = x_3\}$ is called the *graph of f* . If f is a reasonably nice function, then S is what we call a surface. We shall see more of this later. Let us now return to the general case of a function $f: \mathbf{R}^n \rightarrow \mathbf{R}$. The derivative of f is just

a row vector $f'(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \quad \cdots \quad \frac{\partial f}{\partial x_n} \right)$. It is frequently called the *gradient of f* and denoted $\text{grad } f$ or ∇f .

8.2 The Directional Derivative

In the applications of scalar fields it is of interest to talk of the rate of change of the function in a specified direction. Suppose, for instance, the function $T(x, y, z)$ gives the temperature at points (x, y, z) in space, and we might want to know the rate at which the temperature changes as we move in a specified direction. Let $f: \mathbf{R}^n \rightarrow \mathbf{R}$, let $\mathbf{a} \in \mathbf{R}^n$, and let $\mathbf{u} \in \mathbf{R}^n$ be a vector such that $|\mathbf{u}| = 1$. Then the *directional derivative of f at \mathbf{a} in the direction of the vector \mathbf{u}* is defined to be

$$D_{\mathbf{u}} f(\mathbf{a}) = \left. \frac{d}{dt} f(\mathbf{a} + t\mathbf{u}) \right|_{t=0}.$$

Now that we are experts on the Chain Rule, we know at once how to compute such a thing. It is simply

$$D_u f(\mathbf{a}) = \left. \frac{d}{dt} f(\mathbf{a} + t\mathbf{u}) \right|_{t=0} = \nabla f \cdot \mathbf{u}.$$

Example

The surface of a mountain is the graph of $f(x, y) = 700 - x^2 - 5y^2$. In other words, at the point (x, y) , the height is $f(x, y)$. The positive y -axis points North, and, of course, then the positive x -axis points East. You are on the mountain side above the point $(2, 4)$ and begin to walk Southeast. What is the slope of the path at the starting point? Are you going uphill or downhill? (Which!?).

The answers to these questions call for the directional derivative. We know we are at the point $\mathbf{a} = (2, 4)$, but we need a unit vector \mathbf{u} in the direction we are walking. This is,

of course, just $\mathbf{u} = \frac{1}{\sqrt{2}}(1, -1)$. Next we compute the gradient $\nabla f(x, y) = [-2x, -10y]$. At

the point \mathbf{a} this becomes $\nabla f(2, 4) = [-2, -40]$, and at last we have

$$\nabla f \cdot \mathbf{u} = \frac{-2 + 40}{\sqrt{2}} = \frac{38}{\sqrt{2}}.$$

This gives us the slope of the path; it is positive so we are going uphill. Can you tell in which direction the path will be level?

Another Example

The temperature in space is given by $T(x, y, z) = x^2 y + yz^3$. From the point $(1, 1, 1)$, in which direction does the temperature increase most rapidly?

We clearly need the direction in which the directional derivative is largest. The directional derivative is simply $\nabla T \cdot \mathbf{u} = |\nabla T| \cos \theta$, where θ is the angle between ∇T and \mathbf{u} . Anyone can see that this will be largest when $\theta = 0$. Thus T increases most rapidly in

the direction of the gradient of T . Here that direction is $[2xy, x^2 + z^3, 3yz^2]$. At $(1,1,1)$, this becomes $[2, 2, 3]$.

Exercises

1. Find the derivative of $f(x, y, z) = x \log z + 2xy$ at $(1, 2, 1)$ in the direction of the vector $[1, 2, 2]$.
2. Find the derivative of $f(x, y, z) = x \cos y + 3z^3 - xz$ at $(1, \pi, 1)$ in the direction of the vector $[3, -2, 2]$.
3. Find the directions in which $g(x, y) = x^2 y + e^{-xy} \sin y$ increases and decreases most rapidly from the point $(1, 0)$.
4. The surface of a hill is the graph of the equation $z = 1000 + x^2 - x^4 - y^2$. You stand on the hill above the point $(5,3)$ and pour out a glass of water. In which direction will it begin to run? Explain.
5. The position of a particle at time t is given by $\mathbf{r}(t) = 3(t^2 - \sin t)\mathbf{i} + t\mathbf{j} - \cos t\mathbf{k}$, and the position of another particle is $\mathbf{R}(t) = t^2\mathbf{i} + (t^3 + t)\mathbf{j} + \sin t\mathbf{k}$. At time $t = \pi$, what is the rate of change of the distance between the two particles? Are they getting closer to one another, or are they getting farther apart? (Which!) Explain.

8.3 Surface Normals

Let $f: \mathbf{R}^3 \rightarrow \mathbf{R}$ be a function and let c be some constant. Recall that the set $S = \{(x, y, z) \in \mathbf{R}^3 : f(x, y, z) = c\}$ is called a *level set*, or *level surface*, of the function f . Suppose $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ describes a curve in \mathbf{R}^3 that lies on the surface S . This means, of course, that $f(\mathbf{r}(t)) = f(x(t), y(t), z(t)) = c$. Now look at the derivative with respect to t of this equation:

$$\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f \cdot \mathbf{r}'(t) = 0.$$

In other words, the gradient of f and the tangent to the curve are perpendicular. Note there was nothing special about our choice of $\mathbf{r}(t)$; it is *any* curve on the surface. The gradient ∇f is thus *perpendicular*, or *normal* to the surface $f(x, y, z) = c$.

Example

Suppose we want to find an equation of the plane tangent to the surface

$$x^2 + 3y^2 + 2z^2 = 12$$

at the point $(1, -1, 2)$. For an equation of a plane, we need a point \mathbf{a} on the plane and a vector \mathbf{N} normal to the plane. Then the equation we seek is simply $\mathbf{N} \cdot (\mathbf{x} - \mathbf{a}) = 0$, where $\mathbf{x} = (x, y, z)$. In the case at hand, we have a point on the plane: $\mathbf{a} = (1, -1, 2)$. Let's find a normal vector \mathbf{N} . We have just learned that the gradient of $f(x, y, z) = x^2 + 3y^2 + 2z^2$ does the job.

$$\nabla f(x, y, z) = [2x, 6y, 4z],$$

and so $N = f(1, -12) = [2, -6, 8]$. The tangent plane is thus given by the equation $N \cdot (x - a) = 0$, which in this case is

$$2(x - 1) - 6(y + 1) + 8(z - 2) = 0.$$

You should note that the discussion here didn't depend on the dimension of the domain. Thus if $f: \mathbf{R}^2 \rightarrow \mathbf{R}$, then the set $\{(x, y) \in \mathbf{R}^2 : f(x, y) = c\}$ is a level curve of f , and the gradient of f is normal to such a curve.

Combining these results with what we know about the directional derivative, we see that at a point the value of a function increases most rapidly in a direction normal to the level set passing through that point. On a contour map of a portion of the Earth's surface, for example, the steepest path is in the direction normal to the contour lines.

Exercises

6. Find an equation for the plane tangent to the surface $z = x^2 + 2y^2$ at the point $(1, 1, 3)$.
7. Find an equation for the plane tangent to the surface $z = \log(x^2 + y^2)$ at the point $(1, 0, 0)$.
8. Find an equation for the plane tangent to the surface $\cos x - x^2y + e^{xz} + yz = 4$ at the point $(0, 1, 2)$.
9. Find an equation of the straight line tangent to the curve of intersection of the surfaces $x^3 + 3x^2y^2 + y^3 + 4xy - z^2 = 0$ and $x^2 + y^2 + z^2 = 11$ at the point $(1, 1, 3)$.

8.4 Maxima and Minima

Let $f: \mathbf{R}^n \rightarrow \mathbf{R}$. A point \mathbf{a} in the domain of f is called a *local minimum* if there is an open ball $B(\mathbf{a}; r)$ centered at \mathbf{a} such that $f(\mathbf{x}) - f(\mathbf{a}) \geq 0$ for all $\mathbf{x} \in B(\mathbf{a}; r)$. If f is a nice function, then this means the directional derivative $D_{\mathbf{u}}f(\mathbf{a}) = 0$ for all unit vectors \mathbf{u} . In other words, $\nabla f(\mathbf{a}) \cdot \mathbf{u} = 0$. Then it must be true that both $\nabla f(\mathbf{a}) \cdot \mathbf{u} = 0$ and $-\nabla f(\mathbf{a}) \cdot \mathbf{u} = \nabla f(\mathbf{a}) \cdot (-\mathbf{u}) = 0$. This can be so for every \mathbf{u} only if $\nabla f(\mathbf{a}) = \mathbf{0}$. Thus f has a local minimum at a point at which it has a derivative only if the derivative is zero there.

You should guess the definition of a local maximum and see why it must be true that the gradient is zero at such a point. Thus if \mathbf{a} is a local minimum or a local maximum of f , and if f has a derivative at \mathbf{a} , then the derivative $\nabla f(\mathbf{a}) = \mathbf{0}$. You should be aware of the fact that here, just as in Mrs. Turner's elementary calculus class, the converse is not necessarily true. We may have $\nabla f(\mathbf{a}) = \mathbf{0}$ without \mathbf{a} being either a local minimum or a local maximum.

Example

Let us find all local maxima and local minima of the function

$$f(x, y) = x^2 + xy + y^2 + 3x - 3y + 4.$$

Meditate on just how should proceed. This function clearly has a derivative everywhere, so at any local maximum or minimum, this derivative, or gradient, must be zero. So let's begin by finding all points at which $\nabla f(\mathbf{a}) = \mathbf{0}$. In other words, we want (x, y) at which

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0:$$

$$\frac{f}{x} = 2x + y + 3 = 0$$

$$\frac{f}{y} = x + 2y - 3 = 0$$

We are thus faced with the border-line trivial problem of solving the system of equations

$$\begin{aligned} 2x + y &= -3 \\ x + 2y &= 3 \end{aligned}$$

There is just one solution: $(x, y) = (-3, 3)$. Now let us reflect on what we have here. What we have actually found is all the points that *cannot possibly be local minima or maxima*. These are all points *except* $(-3, 3)$. All we know right now is that this point is the only possible candidate. Let's find out what we have by the hammer and tongs method of examining the quantity $f(-3+x, 3+y) - f(-3, 3)$:

$$\begin{aligned} f(-3+x, 3+y) - f(-3, 3) &= f(-3+x, 3+y) - (-5) \\ &= (-3+x)^2 + (-3+x)(3+y) + (3+y)^2 + 3(-3+x) - 3(3+y) + 9 \\ &= x^2 + xy + y^2 = x + \frac{y^2}{2} + \frac{3y^2}{4} \end{aligned}$$

It is therefore clear that $f(-3+x, 3+y) - f(-3, 3) \geq 0$, which means that $(-3, 3)$ is a local minimum.

Exercises

In each of the following, find all local maxima and minima:

10. $f(x, y) = x^2 + 3xy + 3y^2 - 6x + 3y - 6$

11. $f(x, y) = x^2 + xy + 3x + 2y + 5$

12. $f(x, y) = 2xy - 5x^2 - 2y^2 + 4x - 4$

13. $f(x, y) = x^2 + 2xy$

14. $f(x, y) = y - x^2$

8.5 Least Squares

We shall next look at some very simple, yet important, applications in which the location of a minimum value of a function is sought.

Suppose we have a set of n points in the plane, say $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$, and we seek the straight line that "best" fits this collection of points. We first decide what we mean by "best". Let's say we mean the line that minimizes the sum of the squares of the vertical distances from the points to the line. We can describe all nonvertical lines in the world by means of two variables, traditionally called m and b . Thus every such line has the form $y = mx + b$. Our quest is thus for the values of m and b at which the function

$$f(m, b) = \sum_{i=1}^n (mx_i + b - y_i)^2$$

has its minimum value. Knowing these values will give us our line.

We simply apply our vast and growing knowledge of calculus and find where the gradient of f is 0:

$$f = \left(\frac{f}{m}, \frac{f}{b} \right) = \mathbf{0} \quad .$$

Now,

$$\frac{f}{m} = \sum_{i=1}^n 2x_i(mx_i + b - y_i) = 2\left[m \sum_{i=1}^n x_i^2 + b \sum_{i=1}^n x_i - \sum_{i=1}^n x_i y_i \right], \text{ and}$$

$$\frac{f}{b} = \sum_{i=1}^n 2(mx_i + b - y_i) = 2\left[m \sum_{i=1}^n x_i + nb - \sum_{i=1}^n y_i \right].$$

We are thus faced with solving the 2 x 2 linear system

$$\begin{aligned} m \sum_{i=1}^n x_i^2 + b \sum_{i=1}^n x_i &= \sum_{i=1}^n x_i y_i \\ m \sum_{i=1}^n x_i + bn &= \sum_{i=1}^n y_i \end{aligned}$$

Meditate sufficiently to convince yourself that there is always exactly one solution to this system, and continue meditating sufficiently to convince yourself that there must be an honest-to-goodness minimum of the original function at this solution.

Let's have a go at an example. Suppose we have the following table of values:

x	y
0	1
1	2
2	4
3	3.5
4	5

5	4
7	7
8	9
9	12
10	18
12	21
15	29

The linear system for m and b is

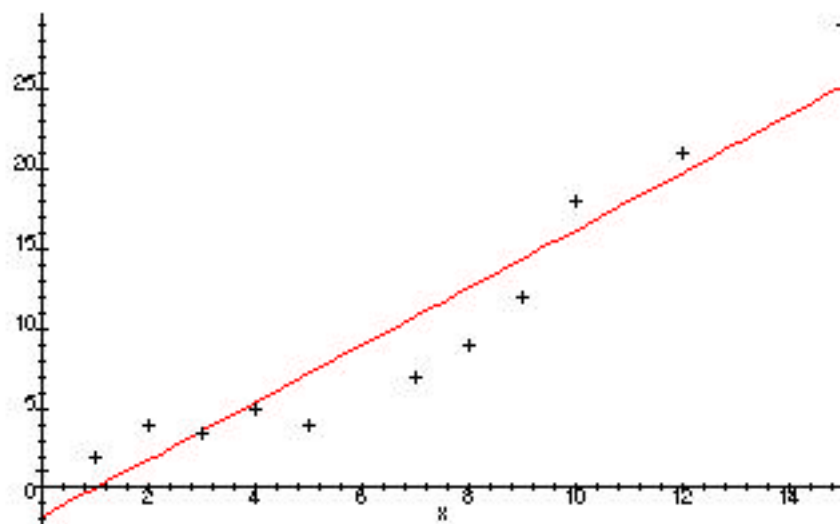
$$718m + 76b = 1156.5$$

$$76m + 12b = 115.5$$

Solving this system gives us $m = \frac{255}{142}$ and $b = -\frac{993}{568}$. In other words, the line that best fits the data in the “sense of least squares” is

$$y = \frac{255}{142}x - \frac{993}{568}$$

Here’s a picture of this line together with the data points:



Looks pretty good!

Exercises

15. Here is a table of Köchel numbers versus year of composition for the compositions of W. A. Mozart. Find the "least squares" straight line approximation to this table and use it to estimate the year in which Mozart's *Sinfonia Concertante in E-flat major* was composed.

Köchel Number	Year composed
1	1761
75	1771
155	1772
219	1775
271	1777
351	1780
425	1783
503	1786
575	1789
626	1791

[This problem is taken from *Calculus and Analytic Geometry (8th Edition)*, by Thomas & Finney.]

16. Find some data somewhere (The *Statistical Abstract of the United States* is a good source of interesting data.), find the least squares linear approximation to the data, and say something intelligent about your results.

8.6 More Maxima and Minima

In real life, one is most likely interested in finding the places at which the largest and smallest values of a function $f: D \rightarrow \mathbf{R}$ occur, rather than in simply finding local maxima and minima. (Here D is a subset of \mathbf{R}^n). To begin, let's think a moment about how we can tell if there is a maximum or minimum value of f on D . First, we suppose that f is continuous—otherwise, anything can happen! Next, what properties of D will insure the existence of a biggest and smallest value of f ? The answer is fairly simple. Certainly D must be a closed subset of \mathbf{R}^n ; consider, for example the function $f: (0,1) \rightarrow \mathbf{R}$ given simply by $f(x) = x$, which has neither a maximum nor a minimum on $D = (0,1)$. Having the domain be closed, however, is not sufficient to guarantee the existence of a maximum and minimum. Consider, for example $f: \mathbf{R} \rightarrow \mathbf{R}$ again with $f: (0,1) \rightarrow \mathbf{R}$ given by $f(x) = x$. The domain \mathbf{R} is certainly closed, but f has neither a maximum nor a minimum. We need also to have the domain be *bounded*. It turns out that for continuous f , if the domain D is both *closed* and *bounded*, then there must necessarily be a maximum and a minimum value for f on D . Let's think a moment about what the candidates for such points are. If the biggest or smallest value of f occurs in the interior of D , then surely the point at which it occurs is a local maximum (or minimum). If f has a gradient there, then the gradient must be 0 . The points at which the largest or smallest values occur must therefore be either i) points in the interior of D at which the gradient of f vanishes, ii) points in the interior at which the gradient of f does not exist, or iii) points in D but not in the interior of D (that is, points on the boundary of D).

Hark back to Mrs. Turner's third grade calculus class. How did you find the maximum value of a function f whose domain D is a closed interval $[a,b] \subset \mathbf{R}$? Recall

found all points in the interior (that is, in the open interval (a,b)) at which the derivative vanishes. You then simply evaluated f at these points, evaluated f at any points in (a,b) at which there is no derivative, evaluated f at the two end points of the interval (in this one dimensional case, the boundary of D is particularly simple.), and then picked out the biggest and smallest numbers you computed. The situation in higher dimensions is a bit more complicated, mostly because the boundary of even a nice domain D is *not* a nice finite set as in the case of an interval, but is an infinite set. Let's look at an example.

Example

A flat circular plate has the shape of the region $\{(x,y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1\}$. The temperature at the point (x,y) on the plate is given by $T(x,y) = x^2 + 2y^2 - x$. Our assignment is to find the hottest and coldest points on the plate. According to our previous discussion, candidates for the hottest and coldest points are all points inside the circular boundary at which the gradient of T is 0 and all points on the boundary. (Note that T has a gradient at all points inside the circle.) First, let's find where among all points (x,y) such that $x^2 + y^2 < 1$, the ones at which $T = (2x - 1, 4y) = \mathbf{0}$. This is easy; it should be clear there is just one such point: $(\frac{1}{2}, 0)$. Now for the more difficult part, finding the candidates on the boundary. Note that the boundary may be described by the vector equation

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}, \text{ where } 0 \leq t < 2\pi.$$

The temperature on this set is then given by

$$T(t) = T(\mathbf{r}(t)), \text{ } 0 \leq t < 2\pi$$

[Here we are abusing the notation, as we have done before, by using the same name for the function $T(x,y)$ and the composition $T(\mathbf{r}(t))$.] We are now faced with the one dimensional problem of finding the maximum and minimum values of a nice differentiable function of one variable on a closed interval. First, we know the endpoints of the interval are candidates: $t = 0$, and $t = 2\pi$. We have at this point added one more point to our list

of candidates: $\mathbf{r}(0) = \mathbf{r}(2\pi) = (1,0)$. Now for candidates inside the interval, we seek places at which the derivative $\frac{dT}{dt} = 0$. From the Chain Rule, we know

$$\frac{dT}{dt} = \nabla T(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (2 \cos t - 1, 4 \sin t) \cdot (-\sin t, \cos t) = 2 \cos t \sin t + \sin t.$$

The equation $\frac{dT}{dt} = 0$ now becomes

$$\begin{aligned} 2 \cos t \sin t + \sin t &= 0, \text{ or} \\ \sin t(2 \cos t + 1) &= 0 \end{aligned}$$

Thus $\sin t = 0$, or $2 \cos t + 1 = 0$. We have, in other words, $y = 0$, or $x = -\frac{1}{2}$. When

$y = 0$, then $x = 1$ or $x = -1$; and when $x = -\frac{1}{2}$, then $y = \frac{\sqrt{3}}{2}$ or $y = -\frac{\sqrt{3}}{2}$. Thus our

new candidates are $(1,0)$, $(-1,0)$, $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$. These together with the one

we have already found, $(\frac{1}{2}, 0)$, make up our entire list of possibilities for the hottest and

coldest points on the plate. All we need do now is to compute the temperature at each of these points:

$$T(\frac{1}{2}, 0) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}.$$

$$T(1, 0) = 1 - 1 = 0$$

$$T(-1, 0) = 1 + 1 = 2$$

$$T(-\frac{1}{2}, \frac{\sqrt{3}}{2}) = T(-\frac{1}{2}, -\frac{\sqrt{3}}{2}) = \frac{1}{4} + \frac{3}{2} + \frac{1}{2} = \frac{9}{4}$$

Finally, we have our answer. The coldest point is $(\frac{1}{2}, 0)$, and the hottest points are

$(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$.

Exercises

17. Find the maximum and minimum value of $f(x, y) = x^2 - xy + y^2 + 4$ on the closed area in the first quadrant bounded by the triangle formed by the lines $x = 0$, $y = 4$, and $y = x$.
18. Find the maximum and minimum values of $f(x, y) = (4y - y^2)\cos x$ on the closed area bounded by the rectangle $1 \leq y \leq 3$, $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$.

8.7 Even More Maxima and Minima

It should be clear now that the really troublesome part of finding maxima and minima is in dealing with the constrained problem; that is, the problem of finding the maxima and minima of a given function on a set of lower dimension than the domain of the function. In the problems of the previous section, we were fortunate in that it was easy to find parametric representations of these sets; in general, this, of course, could be quite difficult. Let's see what we might do about this difficulty.

Suppose we are faced with the problem of finding the maximum or minimum value of the function $f: D \rightarrow \mathbf{R}$, where $D = \{(x, y) \in \mathbf{R}^2 : g(x, y) = 0\}$, where g is a nice function. (In other words, D is a level curve of g .) Suppose $\mathbf{r}(t)$ is a vector description of the curve D . Now then, we are seeking a maximum or minimum of the function $F(t) = f(\mathbf{r}(t))$. At a maximum or minimum, we must have $\frac{dF}{dt} = 0$. (Here g is sufficiently nice to insure that $g(x, y) = 0$ is a closed curve, and so there are no endpoints to worry about.) The Chain Rule tells us that $\frac{dF}{dt} = \nabla f \cdot \mathbf{r}' = 0$. Thus at a maximum or minimum, the gradient of f must be perpendicular to the tangent to $g(x, y) = 0$. But if ∇f is perpendicular to the tangent to the level curve $g(x, y) = 0$, then it must have the

same direction as the normal to this curve. This is just what we need to know, for the gradient of g is normal to this curve. Thus at a maximum or minimum, ∇f and ∇g must "line up". Thus $\nabla f = \lambda \nabla g$, and there is no need actually to know a vector representation \mathbf{r} for $g(x, y) = 0$.

Let's see this idea in action. Suppose we wish to find the largest and smallest values of $f(x, y) = x^2 + y^2$ on the curve $x^2 - 2x + y^2 - 4y = 0$.

Here, we may take $g(x, y) = x^2 - 2x + y^2 - 4y$. Then $\nabla f = 2xi + 2yj$, and $\nabla g = (2x - 2)i + (2y - 4)j$, and our equation $\nabla f = \lambda \nabla g$ becomes

$$\begin{aligned} 2x &= \lambda(2x - 2) \\ 2y &= \lambda(2y - 4) \end{aligned}$$

We obtain a third equation from the requirement that the point (x, y) be on the curve $g(x, y) = 0$. In other words, we need to find all solutions to the system of equations

$$\begin{aligned} 2x &= \lambda(2x - 2) \\ 2y &= \lambda(2y - 4) \\ x^2 - 2x + y^2 - 4y &= 0 \end{aligned}$$

The first two equations become

$$\begin{aligned} x(1 - \lambda) &= 0 \\ y(1 - \lambda) &= 0 \end{aligned}$$

Thus $x = 0$ and $y = 0$. (What about the possibility that $1 - \lambda = 0$?). The last

equation then becomes $\frac{x^2}{(1 - \lambda)^2} - \frac{2x}{1 - \lambda} + \frac{4y^2}{(1 - \lambda)^2} - \frac{8y}{1 - \lambda} = 0$; or,

$$\begin{aligned} x^2 - 2x(1 - \lambda) + 4y^2 - 8y(1 - \lambda) &= 0, \\ x^2 - 2x + 4y^2 - 8y &= 0 \end{aligned}$$

We have two solutions: $x = 0$ and $x = 2$. What do you make of the solution $x = 0$?

These values of x give us two candidates for places at which extrema occur: $x = 0$ and $x = 2$; and $y = 0$ and $y = 4$. Now then $f(0, 0) = 0$, and $f(2, 4) = 4 + 16 = 20$. There

we have them—the minimum value is 0 and it occurs at (0,0); and the maximum value is 20, and it occurs at (2,4).

This method for finding "constrained" extrema is generally called the method of *Lagrange Multipliers*. (The variable λ is called a *Lagrange multiplier*.)

Exercises

19. Use the method of Lagrange multipliers to find the largest and smallest values of $f(x, y) = 4x + 3y$ on the circle $x^2 + y^2 = 1$.
20. Find the points on the ellipse $x^2 + 2y^2 = 1$ at which $f(x, y) = xy$ has its extreme values.
21. Find the points on the curve $x^2 + xy + y^2 = 1$ that are nearest to and farthest from the origin.