

Chapter Nine

The Taylor Polynomial

9.1 Introduction

Let f be a function and let F be a collection of "nice" functions. The approximation problem is simply to find a function $g \in F$ that is "close" to the given function f . There are two issues immediately. How is the collection F selected, and what do we mean by "close"? The answers depend on the problem at hand. Presumably we want to do something to f that is difficult or impossible (This might be something as simple as finding $f(x)$ for some x). The collection F would thus consist of functions to which it is easy to do that which we wish to do to f . Our measure of how close one function is to another would try to reflect the closeness of the results of our operations. Now, what are we talking about here. Suppose, for example, we wish to find $f(x)$. Our collection F of functions should include functions that are easy to evaluate at x , and two functions would be "close" simply if their values are close. We might, for instance, want to evaluate $\sin x$ for all x in some interval I . The collection F could be a collection of second degree polynomials. The approximation problem is then to find elements of F that make the "distance" $\max\{|\sin x - p(x)| : x \in I\}$ as small as possible. Similarly, we might want to find the integral of some function f over an interval I . Here we would want F to consist of functions easily integrated and measure the distance between functions by the difference of their integrals over I . In the previous chapter, we found the "best" straight line approximation to a set of data points. In that case, the collection F consisted of all nonvertical straight lines, and we measured the distance between functions by the sum of the squares of their differences on a specified set of points $\{x_1, x_2, \dots, x_n\}$. You can imagine many other examples.

9.2 The Taylor Polynomial

We look first at a simple but useful problem: Given a nice function $f: D \rightarrow \mathbb{R}$, a point a in the interior of the domain D , and an integer n , find a polynomial p of degree n such that

$$\begin{aligned}
p(a) &= f(a) \\
p'(a) &= f'(a) \\
p''(a) &= f''(a) \\
&\vdots \\
p^{(n)}(a) &= f^{(n)}(a)
\end{aligned}$$

We solve the problem by the Behold Method. Simply verify that

$$p(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

does the job! It is also fairly easy to see that this polynomial is the *only* polynomial of degree n that does the job. Suppose q is also a polynomial with degree $g \leq n$ such that

$$\begin{aligned}
p(a) &= f(a) \\
p'(a) &= f'(a) \\
p''(a) &= f''(a) \\
&\vdots \\
p^{(n)}(a) &= f^{(n)}(a)
\end{aligned}$$

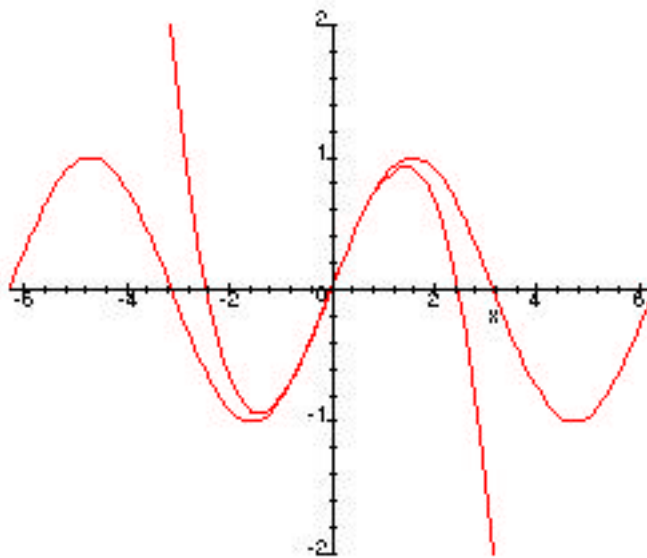
and consider the function $r = p - q$. Note that r is also a polynomial of degree $\leq n$. But

$$r(a) = r'(a) = r''(a) = \dots = r^{(n)}(a) = 0.$$

Or, in other words, r has a zero of order $n + 1$, and the only way this can happen is if $r(x) = 0$ for *all* x . That is, $p(x) = q(x)$ identically.

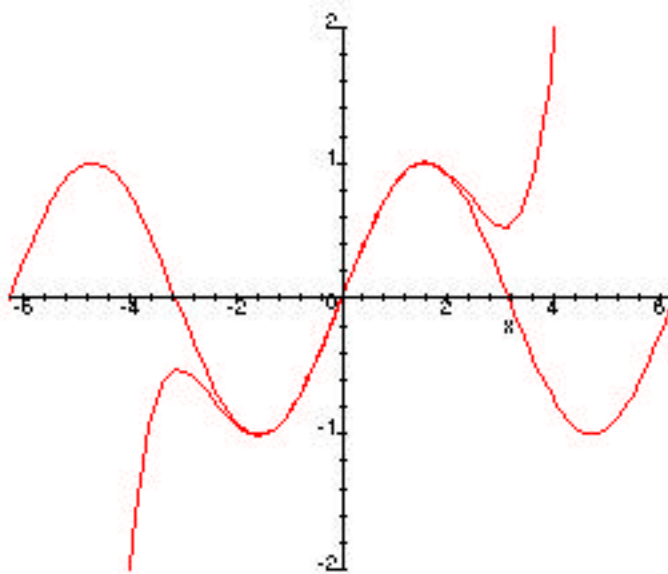
Example

Let $f(x) = \sin x$ and let $a = 0$. Let's find the Taylor polynomial for a few different values of n . For $n = 1$, we have simply $p_1(x) = f(a) + f'(a)(x - a) = \sin 0 + \cos 0(x) = x$. Note that for $n = 2$, we have $p_2(x) = \sin 0 + \cos 0(x) - \sin 0(x^2) = x$, also. Let's take a look at the next Taylor polynomial. Here $p_3(x) = x - \frac{x^3}{6}$. Let's draw some pictures; we'll look at the graph of p_3 and f . We shall use *Maple*.



What we see is that the Taylor polynomial looks like a pretty good approximation as long as we don't get too far away from $a=0$. Let us continue. Convince yourself that $p_4 = p_3$,

and $p_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$. Another picture:



Exercises

1. Find the Taylor polynomial of degree n for $f(x) = e^x$ at $a = 0$.
2. Find the Taylor polynomial of degree n for $f(x) = x^3$ at $a = 1$.
3. Find the Taylor polynomial of degree 3 for $f(x) = \log x$ at $a = 1$.
4. Find the Taylor polynomial of degree n for $f(x) = \sin x$ at $a = 0$.
5. Find the Taylor polynomial of degree 3 for $f(x) = \sqrt{x}$ at $a = 4$.

9.3 Error

Let's see how close the Taylor polynomial is to the function f . To do this, suppose p is the Taylor polynomial of degree n for the function f at a , and consider the function

$$g(t) = f(t) - p(t) - \frac{(t-a)^{n+1}}{(x-a)^{n+1}}(f(x) - p(x)).$$

(We assume $x \neq a$.) Note that $g(a) = g(x) = 0$. Now, from the Mean Value Theorem (or Rolle's Theorem, or whatever.) we know that $g'(\xi_1) = 0$ for some ξ_1 between a and x .

But note also that $g'(a) = f'(a) - p'(a) - \frac{(n+1)(a-a)^n}{(x-a)^{n+1}}(f(x) - p(x)) = 0$. It thus follows

from the Mean Value Theorem that the derivative of g' is zero at some ξ_2 between a and

ξ_1 . Also, $g''(a) = f''(a) - p''(a) - \frac{(n+1)n(a-a)^{n-1}}{(x-a)^{n+1}}(f(x) - p(x)) = 0$. Once again, from

the celebrated Mean Value Theorem, we conclude that $g'''(\xi_3) = 0$ for some ξ_3 between a

and ξ_2 . Continuing in this fashion, we are finally able to conclude that $g^{(n+1)}(\xi) = 0$ for some ξ . Let's see what this looks like.

$$g^{(n+1)}(t) = f^{(n+1)}(t) - p^{(n+1)}(t) - \frac{(n+1)!}{(x-a)^{n+1}}(f(x) - p(x))$$

and so $g^{(n+1)}(x) = 0$ becomes

$$f^{(n+1)}(x) - \frac{(n+1)!}{(x-a)^{n+1}}(f(x) - p(x)) = 0.$$

(Remember, p is a polynomial of degree n , and so $p^{(n+1)}(x) = 0$. From this we obtain an expression for the difference between f and the Taylor polynomial g :

$$f(x) - p(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Example

Remember when in 7th grade physics class, Mr. Crews replaced the sine of a "small" angle by itself? He assured us that for small angles this was just fine. Well, what was going on here? Let's see if our new-found knowledge of Taylor polynomials will help. Observe that $p(x) = x^2$ is simply the Taylor polynomial of degree 2 for $f(x) = \sin x$ at $a = 0$. Using the result just derived, we have that

$$\sin x - x^2 = \frac{-\sin c}{6} x^3.$$

Now, we don't know what c is, but we do know that $|\sin c| \leq 1$; thus

$$|\sin x - x^2| \leq \frac{1}{6} x^3,$$

and we have a precise estimate of the error incurred by substituting x^2 for $\sin x$. Suppose,

for example, that $x = 10^\circ$; then what? Well, $x = \frac{10}{360} 2\pi = \frac{\pi}{18}$. Then the error we get when

we use $\frac{\pi^2}{18}$ instead of $\sin \frac{\pi}{18}$ is estimated by

$$\left| \sin \frac{\pi}{18} - \frac{\pi^2}{18} \right| \leq \frac{1}{6} \left(\frac{\pi}{18} \right)^3 = 0.008862.$$

Now we know exactly what "pretty close" means. For 10 degrees, I guess that's "not too bad."

Exercises

6. a) Find the Taylor polynomial of degree 2 for $f(x) = e^x$ at $a=0$.
- b) Use the result of part a) to find an approximation for \sqrt{e} .
- c) Find as small an upper bound as you can for the difference between your approximation found in part b) and \sqrt{e} .
7. Use the Taylor polynomial found in Exercise 3 to approximate $\log(1.1)$ and find an upper bound for the magnitude of the difference between your approximation and $\log(1.1)$.
8. For what values of x can you replace $\sin x$ by $x - \frac{x^3}{6}$ with an error of magnitude no greater than 3×10^{-4} ?
9. Calculate e with an error of less than 10^{-6} .