Taylor's Theorem

1. **Introduction**. Suppose f is a one-variable function that has $n+1$ derivatives on an interval about the point $x = a$. Then recall from Ms. Turner's class the single variable version of Taylor's Theorem tells us that there is exactly one polynomial p of degree $\leq n$ such that $p(a) = f(a)$, $p'(a) = f'(a)$, $p''(a) = f''(a), \ldots, p^{(n)}(a) = f^{(n)}(a)$. This polynomial is given by

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}}{n!}(x-a)^n$$

We also know the difference between $f(x)$ and $p(x)$:

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1},$$

where ξ is somewhere between a and x.

The polynomial p is called the **Taylor Polynomial** of degree $\leq n$ for f at a.

Before we worry about what the Taylor polynomial might be in higher dimensions, we need to be sure we understand what is a polynomial in more than one dimension. In two dimensions, a polynomial $p(x,y)$ of degree $\leq n$ is a function of the form

$$p(x,y) = \sum_{i,j=0}^{i+j=n} a_{ij}x^i y^j.$$

Thus a polynomial of degree ≤ 2 (perhaps more commonly known as a quadratic) looks like

$$p(x,y) = a_{00} + a_{10}x + a_{01}y + +a_{11}xy + a_{20}x^2 + a_{02}y^2.$$

I hope it easy to guess what one means by a polynomial in three variables, (x,y,z), or indeed, in any number of variables.

Now, how might we extend the idea of the Taylor polynomial of degree $\leq n$ for a function f at a point $\mathbf{a}$? Simple enough. It's a polynomial $p(\mathbf{x})$ of degree $\leq n$ so that

$$\frac{\partial^{i_1+\ldots+i_q} f(a)}{\partial x_1^{i_1} \partial x_2^{i_2} \ldots \partial x_q^{i_q}} = \frac{\partial^{i_1+\ldots+i_q} p(a)}{\partial x_1^{i_1} \partial x_2^{i_2} \ldots \partial x_q^{i_q}},$$

for all $i_1, i_2, \ldots, i_q$ such that $i_1 + i_2 + \ldots + i_q \leq n$.

This looks pretty ferocious in general, so let's see what it says for just two variables. In this case, we have $\mathbf{a} = (a,b)$ and the Taylor polynomial $p(x,y)$ at $\mathbf{a}$ becomes the polynomial such that

$$\frac{\partial^{i+j} f(\mathbf{a})}{\partial^i x \partial^j y} = \frac{\partial^{i+j} p(\mathbf{a})}{\partial^i x \partial^j y},$$

for all $i + j \leq n$.

Example

Let $f(x,y) = \cos(x+y)$, and let $p(x,y) = 1 - \frac{x^2}{2} - xy - \frac{y^2}{2}$. Let's verify that p is the Taylor polynomial of degree ≤ 2 for f at $(0,0)$. He we go.

$$f(0,0) = 1, \text{ and } p(0,0) = 1;$$
$$\frac{\partial f}{\partial x} = -\sin(x+y), \text{ and } \frac{\partial p}{\partial x} = -x - y;$$
$$\frac{\partial f}{\partial y} = -\sin(x+y), \text{ and } \frac{\partial p}{\partial y} = -x - y;$$
$$\frac{\partial^2 f}{\partial x^2} = -\cos(x+y), \text{ and } \frac{\partial^2 p}{\partial x^2} = -1,$$
$$\frac{\partial^2 f}{\partial y^2} = -\cos(x+y), \text{ and } \frac{\partial^2 p}{\partial y^2} = -1,$$
$$\frac{\partial^2 f}{\partial x \partial y} = -\cos(x+y), \text{ and } \frac{\partial^2 p}{\partial x \partial y} = -1.$$

Now it's easy to see that

$$f(0,0) = 0 = p(0,0);$$
$$\frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial p}{\partial x}(0,0);$$
$$\frac{\partial f}{\partial y}(0,0) = 0 = \frac{\partial p}{\partial y}(0,0);$$
$$\frac{\partial^2 f}{\partial x^2}(0,0) = -1 = \frac{\partial^2 p}{\partial x^2}(0,0);$$
$$\frac{\partial^2 f}{\partial y^2}(0,0) = -1 = \frac{\partial^2 p}{\partial y^2}(0,0); \text{ and}$$
$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = -1 = \frac{\partial^2 p}{\partial x \partial y}(0,0).$$

Exercises

1. Verify that the polynomial in the Example is also the Taylor polynomial for f at (0,0) of degree ≤ 3.

2. Let $f(x,y) = \sin(x+y)$.Which Which of the following is the Taylor polynomial of degree ≤ 2 for f at (0,0)? Explain.

a) $p(x,y) = 1 + x^2 + y^2$ b) $p(x,y) = xy$

c) $p(x,y) = x^2 + xy + 2y$ d) $p(x,y) = x + y$

2. **Derivatives**. Prior to finding a general recipe for the Taylor polynomial, we need look at finding higher order derivatives of certain composite functions. Let f be a real-valued function defined on a subset of $\mathbf{R}^q$. Suppose that in a neighborhood of the point $\mathbf{x}$, the function f has a lot of continuous partial derivatives. Define the function g by

$$g(t) = f(\mathbf{a} + t\mathbf{h}),$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_q)$ and $\mathbf{h} = (h_1, h_2, \ldots, h_q)$. We know from the chain rule that $g'(t)$ is given by

$$\begin{aligned} g'(t) &= \nabla f(\mathbf{a} + t\mathbf{h}) \bullet \mathbf{h} \\ &= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_q} \right) \bullet (h_1, h_2, \ldots, h_q) \\ &= \left(h_1 \frac{\partial}{\partial x_1} + h_2 \frac{\partial}{\partial x_2} + \ldots + h_q \frac{\partial}{\partial x_q} \right) f \bigg|_{(\mathbf{a}+t\mathbf{h})} \end{aligned}$$

In keeping with our general practice of restricting ourselves to dimensions one, two, or three, let's look first at the case $q = 2$. As usual, we'll write $\mathbf{x} = (x, y)$ and $\mathbf{h} = (h, k)$. The expression for $g'(t)$ now looks like:

$$g'(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right) f \bigg|_{(\mathbf{x}+t\mathbf{h})}$$

We are now in business, for we have a nice recipe for higher order derivatives of g :

$$g^{(m)}(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^m f \bigg|_{(\mathbf{x}+t\mathbf{h})}$$

For example,

$$\begin{aligned} g''(t) &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^2 f \\ &= \left(h^2 \frac{\partial^2}{\partial x^2} + 2hk\frac{\partial^2}{\partial x \partial y} + k^2 \frac{\partial^2}{\partial y^2} \right) f \\ &= h^2 \frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2} \end{aligned}$$

Example

Suppose $f(x,y) = x^2y^3 + y^2$. Let's find the second derivative of the function

$$g(t) = f(1 + 3t, -2 + t)$$

First,

$$g''(t) = \left(3\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 f$$
$$= 9\frac{\partial^2 f}{\partial x^2} + 6\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}$$

Now, $\frac{\partial f}{\partial x} = 2xy^3$, and $\frac{\partial f}{\partial y} = 3x^2y^2 + 2y$, and so $\frac{\partial^2 f}{\partial x^2} = 2y^3$, $\frac{\partial^2 f}{\partial y \partial x} = 6y^2$, and $\frac{\partial^2 f}{\partial y^2} = 6x^2y + 2$. Thus,

$$g''(t) = 18(-2+t)^3 + 36(-2+t)^2 + 6(1+3t)^2(-2+t) + 2$$

Exercises

3. Let $f(x,y) = xe^y$. Find the derivative of $g(t) = f(1+t, 3-4t)$.

4. Find the second derivative of the function g defined in **Problem 3**.

5. Let $F(u,v) = u^3v + v^2$. Find the second derivative of $R(z) = F(z, 3z)$.

6. Find $g'''(t)$, where g is the function defined in the Example.

3. **The Taylor polynomial**. To find the Taylor polynomial for a function f of several variables at a point **a**, we shall simply apply the one-dimensional results to the function

$$g(t) = f(\mathbf{a} + t\mathbf{h}).$$

Thus,

$$g(t) = \sum_{m=0}^{n} \frac{g^{(m)}(0)}{m!} t^m + \frac{g^{(n+1)}(\xi)}{(n+1)!} t^{n+1},$$

where ξ is a number between 0 and t. Next, substitute $t = 1$ into the above:

$$g(1) = f(\mathbf{a}) = \sum_{m=0}^{n} \frac{g^{(m)}(0)}{m!} + \frac{g^{(n+1)}(\xi)}{(n+1)!}$$

We know the value of $g^{(k)}$ from **Section 2**:

$$f(\mathbf{a}+\mathbf{h}) = \sum_{m=0}^{n} \frac{1}{m!}\left(h_1\frac{\partial}{\partial x_1} + h_2\frac{\partial}{\partial x_2} + \ldots + h_q\frac{\partial}{\partial x_q}\right)^m f(\mathbf{a})$$

$$+\frac{1}{(n+1!}\left(h_1\frac{\partial}{\partial x_1}+h_2\frac{\partial}{\partial x_2}+\ldots+h_q\frac{\partial}{\partial x_q}\right)^{n+1}f(\mathbf{c})$$

The point **c** lies somewhere on the line segment joining **a** and **a** + **h**.
The polynomial

$$p(\mathbf{h}) = p(h_1,h_2,\ldots,h_q) = \sum_{m=0}^{n}\frac{1}{m!}\left(h_1\frac{\partial}{\partial x_1}+h_2\frac{\partial}{\partial x_2}+\ldots+h_q\frac{\partial}{\partial x_q}\right)^{m}f(\mathbf{a})$$

is the Taylor polynomial of degree $\leq n$ for f at **a**; the last term is traditionally called the **error term** or sometimes, the **remainder term**. Actually, if we let $\mathbf{h} = \mathbf{x} - \mathbf{a}$, then $q(\mathbf{x}) = p(\mathbf{x} - \mathbf{a})$ is the thing we called the Taylor polynomial in the first section.

This is pretty fierce looking. Let's look at the two variable case:

$$f(a_1+h,a_2+k) = \sum_{m=0}^{n}\frac{1}{m!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{m}f(a_1,a_2)$$
$$+\frac{1}{(n+1!)}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1}f(c_1,c_2)$$

where (c_1,c_2) is on the line joining (a_1,a_2) and (a_1+h,a_2+k).

Example

Let $f(x,y) = \sin x\sin y$. For $n = 2$ and $\mathbf{a} = (0,0)$, Taylor's polynomial becomes

$$p(h,k) = f(0,0)+h\frac{\partial f}{\partial x}(0,0)+k\frac{\partial f}{\partial y}(0,0)+\frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}(0,0)+hk\frac{\partial^2 f}{\partial x\partial y}(0,0)+\frac{k^2}{2}\frac{\partial^2 f}{\partial y^2}(0,0)$$

We have
$\frac{\partial f}{\partial x} = \cos x\sin y$; $\frac{\partial f}{\partial y} = \sin x\cos y$; $\frac{\partial^2 f}{\partial x^2} = -\sin x\sin y$; $\frac{\partial^2 f}{\partial x\partial y} = \cos x\cos y$; $\frac{\partial^2 f}{\partial y^2} = -\sin x\sin y$.
Thus,

$$p(h,k) = hk.$$

Let's get an estimate for how well this approximates $\sin x\sin y$ near $(0,0)$. We know that

$$|\sin x\sin y - xy| = \left|\frac{1}{3!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^{3}f(\xi,\mu)\right|$$

where (ξ,μ) is one the segment joining (x,y) and the origin. Now,

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^3 f = x^3\frac{\partial^3 f}{\partial x^3}+3x^2y\frac{\partial^3 f}{\partial x^2\partial y}+3xy^2\frac{\partial^3 f}{\partial x\partial y^2}+y^3\frac{\partial^3 f}{\partial x^3}.$$

Next, let's suppose that $|x| \le c$ and $|y| \le c$ for some constant c. Noting that all the partial derivatives in the above expression are simply products of sine and cosines, we can estimate

$$\left|\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^3 f\right| \le 8c^3,$$

and so, at last,

$$|\sin x \sin y - xy| \le \frac{8c^3}{6} = \frac{4}{3}c^3$$

Exercises

7. Find the Taylor polynomial of degree ≤ 1 for $f(x,y) = e^{xy}$ at $(0,0)$.

8. Find the Taylor polynomial of degree ≤ 2 for $f(x,y) = e^{xy}$ at $(0,0)$.

9. Find the Taylor polynomial of degree ≤ 3 for $f(x,y) = e^{xy}$ at $(0,0)$.

10. Find the Taylor polynomial of degree ≤ 1 for $f(x,y) = e^x \cos y$ at $(0,0)$.

11. Use Taylor's Theorem to find a quadratic approximation of $e^x \cos y$ at the origin.

12. Estimate the error in the approximation found in Problem **11** if $|x| \le 0.1$ and $|y| \le 0.1$.