Math 4581 Final Examination Spring 2001 - Solutions

1. In the space of all continuous real-valued functions on the interval \([0,1]\) with the inner product 
\[(f,g) = \int_0^1 f(x)g(x)\,dx,\] 
let \(V\) be the space spanned by \(\{1, x\}\).

a) Find an orthogonal base for \(V\).

Use Gram-Schmidt:

Let \(u_1 = 1\). Then

\[
u_2 = \sqrt{x} - \frac{(u_1, \sqrt{x})}{(u_1, u_1)} u_1 = \sqrt{x} - \frac{\int_0^1 \sqrt{x}\,dx}{\int_0^1 dx} = \sqrt{x} - \frac{2}{3}.
\]

Thus our orthogonal base is

\[\{1, \sqrt{x} - \frac{2}{3}\}\]

b) What is the dimension of \(V\)? Explain.

The above collection is a base for \(V\). It spans, and being orthogonal, it is independent. It contains two items, so the dimension of \(V\) is 2.

c) Find the projection of \(x\) onto \(V\).

\[
\text{proj}(x : u_1, u_2) = \frac{(x, u_1)}{(u_1, u_1)} u_1 + \frac{(x, u_2)}{(u_2, u_2)} u_2.
\]

Now,

\[
(x, u_1) = \int_0^1 x\,dx = \frac{1}{2}
\]

\[
(u_1, u_1) = 1
\]

\[
(x, u_2) = \int_0^1 x\left(\sqrt{x} - \frac{2}{3}\right)\,dx = \frac{1}{15}
\]
\[ (u_2, u_2) = \int_0^1 \left( \sqrt{x} - \frac{2}{3} \right)^2 \, dx = \frac{1}{18} \]

Finally,

\[ \text{proj}(x : u_1, u_2) = \frac{1}{2} + \frac{18}{15} \left( \sqrt{x} - \frac{2}{3} \right) \]
\[ = -\frac{3}{10} + \frac{6}{5} \sqrt{x} \]

e) Find an element \( g \) in \( V \) such that

\[ \int_0^1 [g(x) - x]^2 \, dx \leq \int_0^1 [f(x) - e^{2x}]^2 \, dx \]

for all \( f \) in \( V \).

This is, of course, simply the projection of \( x \) onto \( V \), and we have just found this to be

\[ \text{proj}(x : u_1, u_2) = -\frac{3}{10} + \frac{6}{5} \sqrt{x} \]

2. Let

\[ f(x) = \frac{1}{x^2 + 1}. \]

Let \( c(x) \) be the limit of the Fourier cosine series for \( f \) on the interval \([0, \pi]\); let \( s(x) \) be the limit of the Fourier sine series for \( f \) on \([0, \pi]\); let \( C(x) \) be the limit of the Fourier cosine integral for \( f \); and let \( S(x) \) be the limit of the Fourier sine integral for \( f \). Sketch the graph of each of the functions \( c(x), s(x), C(x), \) and \( S(x) \) on the interval \([-3\pi, 3\pi]\).

For your viewing pleasure, here is a picture of \( f \):

\[ c(x) : \text{even periodic extension of } f \text{ on } [0, \pi]: \]
\( s(x) \): odd periodic extension of \( f \) on \([0, \pi]\):

\[ C(x) \]: even extension of \( f \) (This is simply \( f \) since \( f \) is even.)

\( S(x) \): odd extension of \( f \) to entire real line:

3. a) Find all eigenvalues and corresponding eigenfunctions:

\[ \phi''(x) = -\lambda^2 \phi(x) \]
\[ \phi'(0) = \phi'(\pi) = 0 \]

For \( \lambda \neq 0 \):
\[ \varphi(x) = A \cos \lambda x + B \sin \lambda x \]
\[ \varphi'(x) = \lambda (-A \sin \lambda x + B \cos \lambda x) \]
\[ \varphi'(0) = \lambda B = 0 \]

Thus \( B = 0 \) since \( \lambda \neq 0 \). Then
\[ \varphi'(\pi) = -\lambda A \sin \lambda \pi = 0 \]

Thus, as happens so often, we have \( \lambda_n = 1, 2, 3, \ldots \), with corresponding eigenfunctions \( \varphi_n = \cos \lambda_n x = \cos nx \).

Now, what about \( \lambda = 0 \)? In this case, our differential equation becomes \( \varphi''(x) = 0 \). Thus
\[ \varphi(x) = Ax + B, \quad \varphi'(x) = A \]

Hence \( \varphi'(0) = A = 0 \) means \( A = 0 \). So \( \lambda_0 = 0 \) is also an eigenvalue with eigenfunction \( \varphi_0 = 1 \).

4. Solve:

\[ u_{xx} - u_t = 0, \quad 0 < x < \pi, \quad t > 0 \]
\[ u_x(0, t) = u_x(\pi, t) = 0 \]
\[ u(x, 0) = 5 + \cos 2x, \quad \text{and} \quad u_t(x, 0) = 1. \]

From Problem 3, we know to let
\[ u(x, t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t) \cos nx. \]

The equation becomes
\[ u_{xx} - u_t = \sum_{n=1}^{\infty} \left[ -n^2 a_n(t) - a_n''(t) \right] \cos nx = 0. \]

Then \(-n^2 a_n(t) - a_n''(t) = 0\) gives us \( a_n(t) = a_n \cos nt + b_n \sin nt \) for \( n \geq 1 \) and \( a_0(t) = a_0 t + b_0 \). Hence
\[ u(x, t) = a_0 t + b_0 + \sum_{n=1}^{\infty} \left[ a_n \cos nt + b_n \sin nt \right] \cos nx \]

Now,
\[ u(x, 0) = b_0 + \sum_{n=1}^{\infty} a_n \cos nx = 5 + \cos 2x \]

Thus, \( b_0 = 5, a_2 = 1 \), and \( a_n = 0 \) for all other \( n \).

Next,
\[ u_t(x, 0) = a_0 + \sum_{n=1}^{\infty} b_n \cos nx = 1 \]

This tells us that \( a_0 = 1 \) and \( b_n = 0 \) for all \( n \). The solution to our problems is therefore
\[ u(x, t) = t + 5 + \cos 2t \cos 2x \]

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