Math 4581 Quiz 1 Solution

1. Let $V$ be the usual linear space of all real-valued functions defined on the real line. Which of the following subsets are subspaces of $V$? Be sure and explain your answers.

a) The set of all functions with no more than 2 discontinuities.

This is not a linear space. The functions $f$ and $g$ defined by

$$f(x) = \begin{cases} 
0 & x < 0 \\
1 & 0 \leq x < 1 \\
0 & 1 \leq x
\end{cases}, \text{ and } g(x) = \begin{cases} 
0 & x < 2 \\
1 & 2 \leq x < 3 \\
0 & 3 \leq x
\end{cases}$$

each have exactly two discontinuities, but $f + g$ has more than two discontinuities.

b) The set of all functions with at least 2 discontinuities.

This one is not a linear space. The function $f(x) = 0$ does not have at least 2 discontinuities.

c) The set of all bounded functions.

This one is a linear space. Suppose $f$ and $g$ are bounded; that is, $|f(x)| \leq M$ and $|g(x)| \leq N$ for some constants $M$ and $N$. Then $|f(x) + g(x)| \leq |f(x)| + |g(x)| = M + N$, and for any scalar $a$, we have $|af(x)| = |a||f(x)| \leq |a|M$. Thus both $f + g$ and $af$ are bounded also.

d) The set of all functions $f$ such that $|f(x)| \leq 1$ for all $x$.

This one is not a linear space. $f(x) = 1$ is in the set but $2f(x) = 2$ is not.

e) The set of all functions $f$ such that $f(\pi) = 0$.

This one is a linear space. Suppose $f$ and $g$ are in the set; that is, suppose $f(\pi) = g(\pi) = 0$. Then $(f + g)(\pi) = f(\pi) + g(\pi) = 0 + 0 = 0$, and so $f + g$ is also in the set. Also, if $a$ is a scalar, then $af(\pi) = a0 = 0$, and we have $af$ in the set.

2. Let $V$ be the linear space of all real polynomials together with the inner product

$$\langle p, q \rangle = \int_0^1 xp(x)q(x)dx.$$ 

Find an orthogonal base for the subspace of $V$ consisting of all quadratic polynomials.

The set $\{1, x, x^2\}$ is clearly a base for the subspace of all quadratics. We unleash the Gram-Schmidt procedure to find an orthogonal base $\{\varphi_1, \varphi_2, \varphi_3\}$. First,
\( \varphi_1 = 1. \) Then

\[
\text{proj}(x; \varphi_1) = \frac{(x, 1)}{(1, 1)}
\]

where \( (x, 1) = \int_0^1 x(x \cdot 1)dx = \frac{1}{3} \) and \( (1, 1) = \int_0^1 x(1 \cdot 1)dx = \frac{1}{2} \). Thus,

\[
\varphi_2 = x - \text{proj}(x; \varphi_1) = x - \frac{1 \cdot 2}{3 \cdot 1} = x - \frac{2}{3}.
\]

Next,

\[
\varphi_3 = x^2 - \text{proj}(x^2; \varphi_1, \varphi_2)
\]

\[
\varphi_3 = x^2 - \frac{(x^2, 1)}{(1, 1)} - \frac{(x^2, (x - 2/3))}{(x - 2/3, x - 2/3)}(x - 2/3).
\]

Now,

\[
(x^2, 1) = \int_0^1 x(x^2 \cdot 1)dx = \frac{1}{4}.
\]

\[
(x^2, (x - 2/3)) = \int_0^1 x[x^2(x - 2/3)]dx = \frac{1}{30}
\]

\[
(x - 2/3, x - 2/3) = \int_0^1 x(x - 2/3)^2dx = \frac{1}{36}
\]

Thus,

\[
\varphi_3 = x^2 - \frac{(x^2, 1)}{(1, 1)} - \frac{(x^2, (x - 2/3))}{(x - 2/3, x - 2/3)}(x - 2/3)
\]

\[
= x^2 - \frac{1 \cdot 2}{4 \cdot 1} - \frac{1 \cdot 36}{30 \cdot 1}(x - 2/3)
\]

\[
= x^2 - \frac{3}{4} - \frac{6}{5}(x - 2/3) = x^2 - \frac{6}{5}x + \frac{3}{10}.
\]

3. a) Let \( C(x) \) be the limit of the Fourier cosine series of \( f(x) = \sin x \) on the interval \([0, \pi]\). Sketch the graph of \( C(x) \) on the interval \([-2\pi, 2\pi]\). The Fourier cosine series converges to \( \frac{1}{2} [\hat{f}(x^+) + \hat{f}(x^-)] \) where \( \hat{f} \) is the even periodic extension of \( f(x) = \sin x \). Thus,
b) Let $S(x)$ be the limit of the Fourier sine series of $f(x) = \cos x$ on the interval $[0, \pi]$. Sketch the graph of $S(x)$ on the interval $[-2\pi, 2\pi]$.

The Fourier sine series converges to $\frac{1}{2}[\hat{f}(x +) + \hat{f}(x -)]$ where $\hat{f}$ is the odd periodic extension of $f(x) = \cos x$. Thus,