Chapter Six - Laplace’s Equation

Laplace’s equation, or the potential equation, is $\nabla^2 u = 0$, where the operator $\nabla^2$ is the **Laplacian**, the divergence of the gradient. In rectangular coordinates in two dimensions,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

We are interested in the problem of finding $u$ on a region $R$ such that $\nabla^2 u = 0$ in the interior of $R$ and $u$ is some specified function on the boundary of $R$. This is called the **Dirichlet** problem. We begin with a very special example:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x, y < \pi$$

$$u(0, y) = u(\pi, y) = 0, \text{ and } u(x, 0) = f(x), \ u(x, \pi) = g(x).$$

From all that has gone before, it should be clear why we set

$$u(x, y) = \sum_{n=1}^{\infty} a_n(y) \sin nx.$$ 

This gives us

$$\sum_{n=1}^{\infty} [-n^2 a_n(y) + a''_n(y)] \sin nx = 0,$$

which leads to the ordinary differential equation

$$-n^2 a_n(y) + a''_n(y) = 0.$$ 

From this, we conclude that

$$a_n(y) = a_n \cosh ny + b_n \sinh ny.$$ 

Thus,

$$u(x, y) = \sum_{n=1}^{\infty} [a_n \cosh ny + b_n \sinh ny] \sin nx.$$
The remaining boundary conditions lead to

\[ u(x, 0) = \sum_{n=1}^{\infty} a_n \sin nx = f(x) \], and

\[ u(x, \pi) = \sum_{n=1}^{\infty} [a_n \cosh n\pi + b_n \sinh n\pi] \sin nx = g(x). \]

Hence we need

\[ a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx, \quad \text{and} \]

\[ a_n \cosh n\pi + b_n \sinh n\pi = \frac{2}{\pi} \int_{0}^{\pi} g(x) \, dx, \quad \text{or} \]

\[ b_n = \frac{1}{\sinh n\pi} \left[ \frac{2}{\pi} \int_{0}^{\pi} g(x) \, dx - a_n \cosh n\pi \right]. \]

**Example.** Consider the problem with \( f(x) = x(\pi - x) \) and \( g(x) = 0 \). Then

\[ a_n = \frac{2}{\pi} \int_{0}^{\pi} x(\pi - x) \sin nx \, dx \]

\[ = \frac{4}{\pi n^3} [1 + (-1)^{n+1}] \]

\[ b_n = -a_n \frac{\cosh n\pi}{\sinh n\pi}. \]

Now we have

\[ u(x, y) = \sum_{n=1}^{\infty} \frac{a_n}{\sinh n\pi} [\cosh ny \sinh n\pi - \sinh ny \cosh n\pi] \sin nx \]

\[ = \sum_{n=1}^{\infty} \frac{a_n}{\sinh n\pi} \sin(n(\pi - y)) \sin nx. \]

We can simplify things a bit by observing that \( a_n = 0 \) for \( n \) even. Let \( n = 2k - 1 \). Then

\[ a_{2k-1} = \frac{8}{\pi(2k-1)^3}, \]
and we have

\[ u(x, y) = \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{\sin((2k-1)(\pi - y))}{(2k-1)^3 \sin(2k-1)\pi} \sin(2k-1)x. \]

Here is a picture:

![Picture](image)

**Exercise**

1. Show that the solution to the original problem can be written

\[ u(x, y) = \sum_{n=1}^{\infty} \left[ A_n \frac{\sin(n(\pi - y))}{\sinh n\pi} + B_n \sinh ny \right] \sin nx, \]

where

\[ A_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx dx, \quad B_n = \frac{2}{\pi} \int_{0}^{\pi} g(x) \sin nx dx. \]

2. Find the solution for \( f(x) = 0 \) and \( g(x) = x(\pi - x) \).

Now, what do we do about more realistic boundary conditions; viz., those in which \( u \) does not have to be zero on \( x = 0 \) and \( x = \pi \)? The answer is rather simple. Suppose we want to have \( u(x, 0) = f(x), u(x, \pi) = g(x), \) \( u(0, y) = h(y), \) and \( u(\pi, y) = k(y) \). We first find the solution of the problem in case \( h(y) = k(y) = 0 \). This we have already done. The solution \( v(x, y) \) is given above. Next, we find the solution \( w(x, y) \) of the problem with \( f(x) = g(x) = 0, \) and \( h(y) \) and \( k(y) \) given. Note there is really nothing new here. This is just the previous problem with \( x \) and \( y \) interchanged—we simply turn our heads. The solution \( u \) of the general problem is then

\[ u(x, y) = v(x, y) + w(x, y). \]

**Example.** Let’s solve the general problem with \( f(x) = x^2, \) \( g(x) = 0, \) \( k(y) = (\pi - y)^2, \) and \( h(y) = 0. \) First, \( v.\)

\[ v(x, y) = \sum_{n=1}^{\infty} \left[ A_n \frac{\sin(n(\pi - y))}{\sinh n\pi} \right] \sin nx, \]
\[ A_n = \frac{2}{\pi} \int_0^\pi x^2 \sin nx \, dx = \frac{2}{\pi n^3} [(-1)^{n+1} n^2 \pi^2 + 2((-1)^n - 1)]. \]

Or,

\[ v(x, y) = \frac{2}{\pi} \sum_{n=1}^{40} \frac{(-1)^{n+1} n^2 \pi^2 + 2((-1)^n - 1)}{n^3 \sinh n\pi} \sinh(n(\pi - y)) \sin nx \]

Next,

\[ w(x, y) = \sum_{n=1}^{\infty} \left[ \frac{B_n \sinh nx}{\sinh n\pi} \right] \sin ny, \text{ where} \]

\[ B_n = \frac{2}{\pi} \int_0^\pi (\pi - y)^2 \sin ny \, dy = \frac{2}{\pi} \left[ 2((-1)^n - 1) + n^2 \pi^2 \right] \]

Or,

\[ w(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} (2((-1)^n - 1) + n^2 \pi^2) \frac{\sinh nx \sin ny}{n^3 \sinh n\pi} \]

At last,

\[ u(x, y) = v(x, y) + w(x, y) \]

Here are a couple of views of the graph of \( u \).
Note that it looks fairly nice, except at the corner $(0, \pi)$ of the region. Our problem came with nice continuous boundary conditions, but we then split the problem into two problems each of which has discontinuous boundary conditions. Thus both $v$ and $w$, and hence $u$, are zero at this corner. We are seeing the nasty behavior of the trigonometric series at discontinuities (Gibb’s phenomenon). Could this have been avoided? Yes indeed—we simply replace $u$ by $U = u - V$, where the function $v$ is chosen to insure that $U$ is zero at all four corners of the rectangular region.

How do we find such a $V$? Easy. We let $V(x, y) = a_1 + a_2 x + a_3 y + a_4 xy$, where the $a_i$ are determined so that $V = U$ at the corners. Let’s see how this works with the problem we just completed. In this case, we want

$$
V(0, 0) = a_1 = 0,
V(0, \pi) = a_3 \pi = 0,
V(\pi, \pi) = a_2 \pi + a_4 \pi^2 = 0, \text{ and}
V(\pi, 0) = a_2 \pi = \pi^2.
$$

I hope it is clear that $V(x, y) = \pi x - xy = x(\pi - y)$ does the job. We thus consider the problem

$$
\nabla^2 U = \nabla^2 (u - V) = \nabla^2 u = 0,
U(x, 0) = u(x, 0) - V(x, 0) = x^2 - x\pi
U(\pi, y) = (\pi - y)^2 - \pi(\pi - y) = -y(\pi - y),
U(0, y) = 0 - 0 = 0, \text{ and } U(x, \pi) = 0 - 0.
$$

Study the solution just given and observe $U$ has the same form as the solution to that problem, except we need

$$
A_n = \frac{2}{\pi} \int_0^\pi x(x - \pi) \sin nx \, dx = \frac{4}{n \pi^3} (-1)^n - 1, \text{ and}
B_n = \frac{2}{\pi} \int_0^\pi y(y - \pi) \sin ny \, dy = \frac{4}{n \pi^3} (-1)^n - 1
$$

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B_n = \frac{2}{\pi} \int_0^\pi y(y - \pi) \sin ny \, dy = \frac{4}{n \pi^3} (-1)^n - 1
$$
Thus,

\[ v(x,y) = \frac{4}{\pi} \sum_{n=1}^{\infty} ((-1)^n - 1) \frac{\sinh(n(\pi - y))}{n^3 \sinh n\pi} \sin nx, \text{ and} \]

\[ w(x,y) = \frac{4}{\pi} \sum_{n=1}^{\infty} ((-1)^n - 1) \frac{\sinh nx}{n^3 \sinh n\pi} \sin ny \]

Then

\[ U(x,y) = v(x,y) + w(x,y) \]

\[ = \frac{4}{\pi} \sum_{n=1}^{\infty} ((-1)^n - 1) \frac{\sinh(n(\pi - y))}{n^3 \sinh n\pi} \sin nx + \sinh nx \sin ny. \]

Finally,

\[ u(x,y) = U(x,y) + V(x,y) = U(x,y) + x(\pi - y). \]

Or,

\[ u(x,y) = x(\pi - y) + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{((-1)^n - 1)}{n^3 \sinh n\pi} (\sinh(n(\pi - y)) \sin nx + \sinh nx \sin ny). \]

This is, of course, precisely the same as the solution found before, but this one should have better convergence properties. Let’s take a look at a picture of the first 30 terms of this series:

This one is absolutely gorgeous at the corner \((\pi, 0)!\) Oooh...aahh.]

Exercise
3. Solve
\[ \nabla^2 u = 0, \quad 0 < x, y < \pi \]
\[ u(0, y) = \left( y - \frac{\pi}{2} \right)^2, \quad u(x, \pi) = \left( x - \frac{\pi}{2} \right)^2 \]
\[ u(x, 0) = u(\pi, y) = 0. \]

Consider Laplace’s equation

\[ \nabla^2 u = 0 \]

on the disc of radius \( a \) and centered at the origin. Specifically, consider the problem

\[ \nabla^2 u = 0 \text{ for } x^2 + y^2 \leq c^2, \]
\[ u = f \text{ on the boundary } x^2 + y^2 = c^2. \]

In polar coordinates, the Laplacian operator looks like

\[ \nabla^2 u(r, \theta) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}. \]

Thus we have

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0. \]
\[ u(c, \theta) = g(\theta). \]

I hope it is clear from all that has gone before that we should consider the eigenvalue problem

\[ \frac{d^2 \varphi}{d\theta^2} = -\lambda^2 \varphi \]
\[ \varphi(\pi) = \varphi(-\pi), \text{ and} \]
\[ \varphi'(\pi) = \varphi'(-\pi) \]

From our vast knowledge of Sturm-Liouville problems, we know what to expect. Let’s see what we get.

\[ \varphi(\theta) = A \cos \lambda \theta + B \sin \lambda \theta \]

and so our boundary conditions become
\[ A \cos \lambda \pi + B \sin \lambda \pi = A \cos(-\lambda \pi) + B \sin(-\lambda \pi), \] and
\[ \lambda [-A \sin \lambda \pi + B \cos \lambda \pi] = \lambda [A \sin(-\lambda \pi) - B \cos(-\lambda \pi)]. \]

Or,
\[ 2B \sin \lambda \pi = 0 \]
\[ \lambda A \sin \lambda \pi = 0 \]

A moment’s reflection should convince you that we obtain eigenvalues \( \lambda_n^2 = n^2 \) for \( n = 0, 1, 2, \ldots \). Corresponding to the eigenvalue \( \lambda_0^2 = 0 \), we have the eigenfunction \( \phi(\theta) = 1 \), and corresponding to each eigenvalue \( \lambda_n^2 = n^2 \), we have two independent eigenfunctions \( \cos n\theta \) and \( \sin n\theta \).

With
\[ u(r, \theta) = a_0(r) + \sum_{n=1}^{\infty} [a_n(r) \cos n\theta + \beta_n(r) \sin n\theta] \]
we have
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \\
= \frac{1}{r} \frac{d}{dr} (\alpha_0'(r)) + \sum_{n=1}^{\infty} \left[ \frac{1}{r} \frac{d}{dr} (r\alpha_n'(r)) \cos n\theta + \frac{1}{r} \frac{d}{dr} (r\beta_n'(r)) \sin n\theta \right. \\
- \left. n^2 \frac{a_n(r)}{r^2} \cos n\theta - n^2 \frac{\beta_n(r)}{r^2} \sin n\theta \right] \\
= 0.
\]

Hence,
\[
\frac{1}{r} \frac{d}{dr} (\alpha_0'(r)) \\
+ \sum_{n=1}^{\infty} \left[ \frac{1}{r} \frac{d}{dr} (r\alpha_n'(r)) - n^2 \frac{a_n(r)}{r^2} \right] \cos n\theta + \left( \frac{1}{r} \frac{d}{dr} (r\beta_n'(r)) - n^2 \frac{\beta_n(r)}{r^2} \right) \sin n\theta \\
= 0.
\]

This gives us the differential equations
\[
\frac{1}{r} \frac{d}{dr} (r\alpha_0'(r)) = 0,
\]
\[
\frac{1}{r} \frac{d}{dr} (r\alpha_n'(r)) - \frac{1}{r^2} n^2 a_n(r) = 0, \text{ and}
\]
\[
\frac{1}{r} \frac{d}{dr} (r\beta_n'(r)) - \frac{1}{r^2} n^2 \beta_n(r) = 0.
\]
The first one is easy: \( r' = 0 \). Thus, \( r = A \log r + B \). The requirement that the solution be nice at \( r = 0 \) means that \( A \) must be 0. Thus \( a_0 = \text{constant} = a_0 \). Next,

\[
\frac{1}{r} \frac{d}{dr}(r \alpha_n'(r)) - \frac{1}{r^2} n^2 \alpha_n(r) = 0 \becomes
r^2 \alpha_n''(r) + r \alpha_n'(r) - n^2 \alpha_n(r) = 0.
\]

This, as you no doubt remember from Mrs. Turner’s calculus class, is a so-called Cauchy-Euler equation, all solutions of which are

\[ a_n(r) = Ar^n + Br^{-n}. \]

Again, the solutions must be nice at \( r = 0 \), and so \( B = 0 \), and our solutions are

\[ a_n(r) = a_n r^n. \]

In exactly the same way, we get

\[ \beta_n(r) = b_n r^n. \]

Putting it all together gives us

\[
u(r, \theta) = a_0 + \sum_{n=1}^{\infty} [a_n r^n \cos n\theta + b_n r^n \sin n\theta].\]

The condition \( u(c, \theta) = g(\theta) \) becomes

\[
g(\theta) = a_0 + \sum_{n=1}^{\infty} [a_n c^n \cos n\theta + b_n c^n \sin n\theta].\]

Thus,

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) d\theta, \quad \text{and} \quad a_n = \frac{1}{\pi c^n} \int_{-\pi}^{\pi} g(\theta) \cos n\theta d\theta.\]
Example. Suppose $c = 1$ and $g(\theta) = \theta^2$. Then

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta^2 d\theta = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta^2 \cos n\theta d\theta = \frac{4(-1)^n}{n^2}, \text{ and}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta^2 \sin n\theta d\theta = 0$$

Hence,

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} [a_n r^n \cos n\theta + b_n r^n \sin n\theta]$$

$$u(r, \theta) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{30} \frac{(-1)^n}{n^2} r^n \cos n\theta$$

Here is a picture:

Exercises

4. a) Show that the value of $u$ at the center of the disc, $u(0, \theta)$, is the average of the values of $u$ on the boundary of the disc.

b) Show that the value of $u$ at the center of the disc, $u(0, \theta)$, is the average of the values of $u$ on any circle $r = a \leq c$. 
5. a) Use the result of Problem 4 to show that if $\nabla^2 u = 0$ on some region $R$, then the maximum value of $u$ occurs on the boundary of $u$ only if $u$ is constant on $R$.

b) Show that if $\nabla^2 u = 0$ on some region $R$, then the minimum value of $u$ occurs on the boundary of $u$ only if $u$ is constant on $R$. 