Chapter Seven - Vibrating Strings

The displacement $u$ of a string is described by the equation

$$\frac{\partial}{\partial x} \left( T(x) \frac{\partial u}{\partial x} \right) - \rho(x) \frac{\partial^2 u}{\partial t^2} = 0,$$

where $T(x)$ is the tension and $\rho(x)$ is the density. We have already seen this in the hanging chain problem—there the tension is proportional to $x$ and the density is constant. Let’s go back to the simpler problem of a uniform string fixed at the ends $x = 0$ and $x = \pi$. In this case the tension and the density are both constant: say $T(x) = T$ and $\rho(x) = \rho$. Then

$$u_{xx} - \frac{\rho}{T} u_{tt} = 0, 0 < x < \pi$$

$$u(0,t) = u(\pi,t) = 0, \text{ and}$$

$$u(x,0) = f(x), u_t(x,0) = g(x).$$

From our vast knowledge of eigenvalue problems we know to let $u = \sum_{n=1}^{\infty} a_n(t) \sin nx$, which gives us

$$\sum_{n=1}^{\infty} \left[ -n^2 a_n(t) - \frac{\rho}{T} a_n''(t) \right] \sin x = 0.$$

Thus

$$a_n''(t) + n^2 \frac{T}{\rho} a_n = 0,$$

which has solutions

$$a_n(t) = a_n \cos nvt + b_n \sin nvt, \text{ where}$$

$$v = \sqrt{\frac{T}{\rho}}.$$

Hence,

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos nvt + b_n \sin nvt) \sin nx.$$

From the initial conditions, we know
\[ a_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx, \quad \text{and} \quad b_n = \frac{2}{\pi n v} \int_0^\pi g(x) \sin nx \, dx. \]

Observe that the solution \( u \) is periodic in \( t \): \( u(x, t) = u(x, t + 2\pi/v) \)

**Example.** Suppose \( v = 1 \), \( g(x) = 0 \), and 

\[ f(x) = \begin{cases} 
  x/2 & 0 \leq x \leq \pi/2 \\
 -(x - \pi)/2 & 1/2 < x \leq 1 
\end{cases} \]

Then 

\[ a_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx = \frac{2 \sin \frac{\pi}{2}}{\pi n^2}, \]

and \( b_n = 0 \). Hence, 

\[ u(x, t) = \frac{2}{\pi} \sum_{n=1}^\infty \frac{\sin \frac{n\pi}{2}}{n^2} \cos nt \sin nx. \]

Let’s see what this looks like for a sequence of values of time \( t \).

\[ t = 0 \]

\[ t = \pi/6 \]
Suppose all the terms in the series for \( u \) save the one for \( n = 1 \) are zero. The solution then is simply

\[
  u(x, t) = (a_1 \cos vt + b_1 \sin vt) \sin x \\
  = A \cos (vt + \varphi) \sin x.
\]

In this oscillation, the string always has the shape of a single arch of the sine curve and vibrates with the radian frequency \( v \):
The solution for all $n = 0$ except $n = 2$ has the form $A \cos(2\nu t + \varphi) \sin 2x$. Here the string has the shape of $\sin 2x$ and oscillates with a frequency $2\nu$, or twice the frequency of the previous solution:

Musically, this oscillation would sound an octave higher than the first one. Convince yourself that for $n = 3$, the string would vibrate with frequency $3\nu$:

What is the musical interval between this one and the previous one?

The solutions like these in which all but one of values of $n$ are zero are called **vibration modes**. Thus every solution is a superposition, or "sum", of these modes of vibration.

Let’s consider again the solution in which $g(x) = 0$. Then

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos n\nu t \sin nx.$$
Drawing upon our vast knowledge of trigonometry, we see that

\[ \cos nvt \sin nx = \frac{1}{2} [\sin(n(x + vt)) + \sin(n(x - vt))] . \]

Hence,

\[ u(x, y) = \frac{1}{2} \sum_{n=1}^{\infty} a_n [\sin(n(x + vt)) + \sin(n(x - vt))] \]

\[ = \frac{1}{2} \left( \sum_{n=1}^{\infty} a_n \sin(n(x + vt)) + \sum_{n=1}^{\infty} a_n \sin(n(x - vt)) \right) . \]

Now we know that for most all \( x \)

\[ \tilde{f}(x) = \sum_{n=1}^{\infty} a_n \sin nx, \]

where \( \tilde{f} \) is the odd periodic extension of \( f \). Thus

\[ \tilde{f}(x + vt) = \sum_{n=1}^{\infty} a_n \sin(n(x + vt)), \text{ and } \tilde{f}(x - vt) = \sum_{n=1}^{\infty} a_n \sin(n(x - vt)), \]

and our solution is

\[ u(x, t) = \frac{1}{2} \left( \tilde{f}(x + vt) + \tilde{f}(x - vt) \right) . \]

This is special case of what is called D’Alembert’s formula of the wave equation. In case \( g \) is not zero, there is the full version of D’Alembert’s solution:

\[ u(x, t) = \frac{1}{2} \left( \tilde{f}(x + vt) + \tilde{f}(x - vt) \right) + \frac{1}{2v} \int_{x-vt}^{x+vt} g(s)ds . \]