Chapter Seven - Complete Pseudometric Spaces

**Definition.** A sequence \((x_n)\) in a pseudometric space \((X,d)\) is a **Cauchy sequence** if for every \(\varepsilon > 0\), there is an integer \(N\) so that \(d(x_n,x_m) < \varepsilon\) for all \(n, m \geq N\).

**Proposition 7.1.** Every convergent sequence in a pseudometric space is a Cauchy sequence.

**Example 7.2.** Let \(X\) be the irrational numbers with the usual pseudometric inherited from the space of real numbers. Then \((\sqrt{2}/n)\) is a Cauchy sequence that has no limit.

**Definition.** A pseudometric space in which every Cauchy sequence has a limit is a **complete pseudometric space**.

**Proposition 7.3.** Every sequence in a compact space has a cluster point.

**Proposition 7.4.** A Cauchy sequence with a cluster point converges.

**Theorem 7.5.** Every compact pseudometric space is complete.

**Theorem 7.6.** A closed subspace of a complete pseudometric space is complete.

**Theorem 7.7.** A complete subspace of a metric space is closed.

**Example 7.8.** Let \(X\) be the plane with the pseudometric \(d(x,y) = |x_1 - y_1|\), where \(x = (x_1,x_2)\) and \(y = (y_1,y_2)\). Then the set \(S = \{(t,0) : 0 \leq t \leq 1\}\) is a complete subspace of \((X,d)\), but \(clS = \{(x_1,x_2) : 0 \leq x_1 \leq 1,\text{ and } x_2 \text{ is real}\} \neq S\).

**Definition.** A function \(f : (X,d) \to (Y,\rho)\) from one pseudometric space into another is a **contraction map** if there is a real number \(k < 1\) such that \(\rho(f(x),f(y)) \leq kd(x,y)\) for all \(x, y \in X\).

**Proposition 7.9.** Let \(f : X \to X\) be a contraction from a pseudometric space into itself. Let \(x_0 \in X\) and for each positive integer \(n\), let \(x_n = f(x_{n-1})\). Then the sequence \((x_n)\) is a Cauchy sequence.

**Proposition 7.10.** Let \(f : (X,d) \to (X,d)\) be a contraction map from a complete pseudometric space into itself. Let \(x_0 \in X\), and for each positive integer \(n\), let \(x_n = f(x_{n-1})\). The sequence \((x_n)\) converges, and for any limit \(z\) of the sequence, \(d(z,f(z)) = 0\).

**Theorem 7.11.** Let \(f : (X,d) \to (X,d)\) be a contraction map from a complete metric space into itself. Then there is exactly one point \(z \in X\) such that \(f(z) = z\). Moreover, if \(x_0\) is any point in \(X\) and for each positive integer \(n\), we define \(x_n = f(x_{n-1})\), then the sequence \((x_n)\) converges to \(z\).
Note. This is the celebrated Banach Fixed Point Theorem.

**Theorem 7.12.** Let $X$ be a complete pseudometric space, and let $D = \{D_n : n \in \mathbb{Z}_+\}$ be a countable collection of open dense subsets of $X$. Then $\bigcap D$ is a dense subset of $X$.

**Proposition 7.13.** A subset $A$ of a topological space $X$ is open and dense if and only if $X \setminus A$ is closed and has empty interior.

**Theorem 7.14.** Let $X$ be a complete pseudometric space, and let $F = \{F_n : n \in \mathbb{Z}_+\}$ be a countable collection of closed sets each of which has empty interior. Then $\bigcup F$ has empty interior.

Note. Theorems 7.12 and 7.14 are versions of the famous Baire Category Theorem.