Chapter Eight - Linear Spaces

Definition. A linear space $X$ is a set for which are defined an addition $+$ making $X$ an abelian group, and multiplication by scalars, satisfying the distributive laws $t(x + y) = tx + ty$ and $(s + t)x = sx + tx$, where $t$ and $s$ are scalars, $x, y \in X$, and satisfying $(st)x = s(tx)$ and $1x = x$.

Note. ” Scalars” means complex numbers unless otherwise indicated.

Definition. A linear subspace of a linear space is a subset which with the same operations and scalars is a linear space.

Definition. A function $f : X \to Y$ from one linear space into another is a linear function if $f(x + y) = f(x) + f(y)$ and $f(tx) = tf(x)$ for all scalars $t$ and all $x, y \in X$.

Definition. A set $M \subset X$, a linear space, is a linear variety if $M = x_0 + M_0 = \{x_0 + y : y \in M_0\}$ for some $x_0 \in X$ and some linear subspace $M_0$.

Definition. A set $C \subset X$ is convex if $tx + (1 - t)y \in C$ for all $x, y \in C$ and $0 \leq t \leq 1$.

Proposition 8.1. A linear variety is convex.

Proposition 8.2. Suppose $X$ is a linear space, $a \in X$, $a \neq 0$, and $M = \{ta : \text{all } t\}$. Then $M$ is a linear subspace. [M is traditionally called a straight line through 0.]

Definition. Linear subspaces $M$ and $N$ of a linear space $X$ are said to be complementary if $M \cap N = \{0\}$ and $X = M + N$.

Note. If $A$ and $B$ are subsets of a linear space, $A + B = \{a + b : a \in A$ and $b \in B\}$.

Theorem 8.3. Subspaces $M$ and $N$ of a linear space $X$ are complementary if and only if each $x \in X$ can be expressed uniquely as $x = m + n$, where $m \in M$ and $n \in N$.

Definition. A linear subspace complementary to a straight line through 0 is known as a hyperplane through 0.

Definitions. A linear variety $M = x_0 + M_0$ is a straight line if $M_0$ is a straight line through 0, and is a hyperplane if $M_0$ is a hyperplane through 0.

Theorem 8.4. A linear subspace $M_0$ of a space $X$ is a hyperplane through 0 if and only if there is a nonconstant linear function $f : X \to S$ from $X$ into the scalars such that $M_0 = f^{-1}(0)$. 
Note. A linear function from a linear space into the scalars is frequently called a linear functional.

Corollary 8.5. A linear variety \( M \subset X \) is a hyperplane if and only if there is a nonconstant linear function \( f : X \to S \) from \( X \) into the scalars such that \( M = f^{-1}(t) \) for some scalar \( t \).

**Theorem 8.6.** Suppose \( N_0 \) is a linear subspace of the linear space \( X \), and \( M_0 \) is a hyperplane through \( 0 \). If \( M_0 \subset N_0 \), then either \( M_0 = N_0 \) or \( N_0 = X \).

Definition. Suppose \( X \) is a linear space that is also a topological space, and suppose the scalars are endowed with the usual topology. Then \( X \) is a linear topological space if the functions \( F : X \times X \to X \), and \( G : S \times X \to X \) given by \( F(x,y) = x + y \) and \( G(t,x) = tx \) are both continuous.

**Proposition 8.7.** Let \( M_0 \) be a linear subspace of a linear topological space \( X \), and let \( x_0 \in X \). Then the function \( f : M_0 \to M = x_0 + M_0 \) given by \( f(m) = x_0 + m \) is a homeomorphism.

**Theorem 8.8.** In a linear topological space, the closure of a linear subspace is a linear subspace.

**Corollary 8.9.** In a linear topological space, the closure of a linear variety is a linear variety.

**Theorem 8.10.** Suppose \( M \) is a hyperplane in a linear topological space. If \( M \) is not closed, it is dense.

**Theorem 8.11.** Suppose \( M \) is a hyperplane in a linear topological space \( X \), and suppose \( f : X \to S \) is a linear function into the scalars such that \( M = f^{-1}(t) \) for some scalar \( t \). Then \( M \) is closed if and only if \( f \) is continuous.