1. Find a function $g$ the gradient of which is

$$\mathbf{F}(x,y,z) = (e^{xy} + 2xy + 3x^2)\mathbf{i} + (xze^{xy} + x^2 + 2yz + 3)\mathbf{j} + (xye^{xy} + y^2 + 2z)\mathbf{k},$$

or explain carefully why there is no such $g$.

If $\nabla g = \mathbf{F}$, then

$$\frac{\partial g}{\partial x} = e^{xy} + 2xy + 3x^2.$$

Thus,

$$g(x,y,z) = xe^{xy} + x^2y + x^3 + h(y,z).$$

To find the function $h(y,x)$, differentiate with respect to $y$:

$$\frac{\partial g}{\partial y} = xze^{xy} + x^2 + \frac{\partial h}{\partial y}(y,z).$$

But we know also that

$$\frac{\partial g}{\partial y} = xze^{xy} + x^2 + 2yz + 3.$$

Thus,

$$xze^{xy} + x^2 + \frac{\partial h}{\partial y}(y,z) = xze^{xy} + x^2 + 2yz + 3, \text{ or } \frac{\partial h}{\partial y}(y,z) = 2yz + 3.$$

Integrate to find $h$:

$$h(y,z) = y^2z + 3y + k(z).$$

Put this into the expression we found for $g$:

$$g(x,y,z) = xe^{xy} + x^2y + x^3 + h(y,z) = xe^{xy} + x^2y + x^3 + y^2z + 3y + k(z).$$

To find $k(z)$, differentiate with respect to $z$:

$$\frac{\partial g}{\partial z} = xye^{xy} + y^2 + k'(z),$$

and we know,

$$\frac{\partial g}{\partial z} = xye^{xy} + y^2 + 2z.$$

Thus,

$$xye^{xy} + y^2 + k'(z) = xye^{xy} + y^2 + 2z, \text{ or } k'(z) = 2z.$$

At last, $k(z) = z^2$, and we have
\[
g(x,y,z) = xe^{xyz} + x^2 y + x^3 + y^2 z + 3y + k(z)
= xe^{xyz} + x^2 y + x^3 + y^2 z + 3y + z^2.
\]

2. Find the vector line integral of \( F = xy \mathbf{i} + z \mathbf{j} + (x^2 + z^2) \mathbf{k} \) from \( (0,0,0) \) to \( (1,1,1) \) along the path \( P \) which consists of the curve \( y = x^2 \) from \( (0,0,0) \) to \( (1,1,0) \) together with the straight line from \( (1,1,0) \) to \( (1,1,1) \).

We’ll integrate first from \( (0,0,0) \) to \( (1,1,0) \) along the curve \( y = x^2 \). A vector description of this path, call it \( P_1 \), is simply

\[
\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j}, \quad \text{for } 0 \leq t \leq 1.
\]

Now, \( \mathbf{r}'(t) = \mathbf{i} + 2t \mathbf{j} \), and so

\[
\int_{P_1} F \cdot d\mathbf{r} = \int_0^1 F(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt
= \int_0^1 (t^3 \mathbf{i} + t^2 \mathbf{k}) \cdot (\mathbf{i} + 2t \mathbf{j})\,dt
= \int_0^1 t^3\,dt = \frac{1}{4}.
\]

Next, integrate along the straight line from \( (1,1,0) \) to \( (1,1,1) \):

A vector description for the line, call it \( P_2 \), is just \( \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t \mathbf{k} \), for \( 0 \leq t \leq 1 \). So,

\[
\int_{P_2} F \cdot d\mathbf{r} = \int_0^1 F(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt
= \int_0^1 [\mathbf{i} + \mathbf{j} + (1 + t^2) \mathbf{k}] \cdot \mathbf{k} \,dt
= \int_0^1 (1 + t^2)\,dt = 1 + \frac{1}{3} = \frac{4}{3}.
\]

Finally,

\[
\int_P F \cdot d\mathbf{r} = \int_{P_1} F \cdot d\mathbf{r} + \int_{P_2} F \cdot d\mathbf{r}
= \frac{1}{4} + \frac{4}{3} = \frac{19}{12}.
\]