Chapter Eleven

Argument Principle

11.1. Argument principle. Let $C$ be a simple closed curve, and suppose $f$ is analytic on $C$. Suppose moreover that the only singularities of $f$ inside $C$ are poles. If $f(z) \neq 0$ for all $z \in C$, then $\Gamma = f(C)$ is a closed curve which does not pass through the origin. If

$$\gamma(t), \ \alpha \leq t \leq \beta$$

is a complex description of $C$, then

$$\zeta(t) = f(\gamma(t)), \ \alpha \leq t \leq \beta$$

is a complex description of $\Gamma$. Now, let’s compute

$$\int_\gamma \frac{f'(z)}{f(z)} \, dz = \int_\alpha^\beta \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t) \, dt.$$ 

But notice that $\zeta'(t) = f'(\gamma(t))\gamma'(t)$. Hence,

$$\int_\gamma \frac{f'(z)}{f(z)} \, dz = \int_\alpha^\beta \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t) \, dt = \int_\alpha^\beta \frac{\zeta'(t)}{\zeta(t)} \, dt$$

$$= \int_\Gamma \frac{1}{\zeta} \, d\zeta = n2\pi i,$$

where $|n|$ is the number of times $\Gamma$ ”winds around” the origin. The integer $n$ is positive in case $\Gamma$ is traversed in the positive direction, and negative in case the traversal is in the negative direction.

Next, we shall use the Residue Theorem to evaluate the integral $\int_\gamma \frac{f'(z)}{f(z)} \, dz$. The singularities of the integrand $\frac{f'(z)}{f(z)}$ are the poles of $f$ together with the zeros of $f$. Let’s find the residues at these points. First, let $Z = \{z_1, z_2, \ldots, z_K\}$ be set of all zeros of $f$. Suppose the order of the zero $z_j$ is $n_j$. Then $f(z) = (z-z_j)^{n_j}h(z)$ and $h(z_j) \neq 0$. Thus,

$$\frac{f'(z)}{f(z)} = \frac{(z-z_j)^{n_j}h'(z) + n_j(z-z_j)^{n_j-1}h(z)}{(z-z_j)^{n_j}h(z)}$$

$$= \frac{h'(z)}{h(z)} + \frac{n_j}{z-z_j}.$$
Then
\[
\phi(z) = (z - z_j) \frac{f'(z)}{f(z)} = (z - z_j) \frac{h'(z)}{h(z)} + n_j,
\]
and
\[
\text{Res}_{z=z_j} \frac{f'}{f} = n_j.
\]

The sum of all these residues is thus
\[
N = n_1 + n_2 + \ldots + n_K.
\]

Next, we go after the residues at the poles of \( f \). Let the set of poles of \( f \) be \( P = \{ p_1, p_2, \ldots, p_J \} \). Suppose \( p_j \) is a pole of order \( m_j \). Then
\[
h(z) = (z - p_j)^{m_j} f(z)
\]
is analytic at \( p_j \). In other words,
\[
f(z) = \frac{h(z)}{(z - p_j)^{m_j}}.
\]

Hence,
\[
\frac{f'(z)}{f(z)} = \frac{(z - p_j)^{m_j} h'(z) - m_j (z - p_j)^{m_j - 1} h(z)}{(z - p_j)^{2m_j}} + \frac{(z - p_j)^{m_j}}{h(z)}
\]
\[
= \frac{h'(z)}{h(z)} - \frac{m_j}{(z - p_j)^{m_j}}.
\]

Now then,
\[
\phi(z) = (z - p_j)^{m_j} \frac{f'(z)}{f(z)} = (z - p_j)^{m_j} \frac{h'(z)}{h(z)} - m_j,
\]
and so
\[
\text{Res}_{z=p_j} \frac{f'}{f} = \phi(p_j) = -m_j.
\]

The sum of all these residues is
\[
-P = -m_1 - m_2 - \ldots - m_J
\]
Then,

\[ \oint_C \frac{f'(z)}{f(z)} \, dz = 2\pi i (N - P); \]

and we already found that

\[ \oint_C \frac{f'(z)}{f(z)} \, dz = n2\pi i, \]

where \( n \) is the "winding number", or the number of times \( \Gamma \) winds around the origin—\( n > 0 \) means \( \Gamma \) winds in the positive sense, and \( n \) negative means it winds in the negative sense. Finally, we have

\[ n = N - P, \]

where \( N = n_1 + n_2 + \ldots + n_K \) is the number of zeros inside \( C \), counting multiplicity, or the order of the zeros, and \( P = m_1 + m_2 + \ldots + m_J \) is the number of poles, counting the order. This result is the celebrated argument principle.

**Exercises**

1. Let \( C \) be the unit circle \( |z| = 1 \) positively oriented, and let \( f \) be given by

\[ f(z) = z^3. \]

How many times does the curve \( f(C) \) wind around the origin? Explain.

2. Let \( C \) be the unit circle \( |z| = 1 \) positively oriented, and let \( f \) be given by

\[ f(z) = \frac{z^2 + 2}{z^3}. \]

How many times does the curve \( f(C) \) wind around the origin? Explain.

3. Let \( p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 \), with \( a_n \neq 0 \). Prove there is an \( R > 0 \) so that if \( C \) is the circle \( |z| = R \) positively oriented, then

\[ \oint_C \frac{p'(z)}{p(z)} \, dz = 2n\pi i. \]

4. Suppose \( f \) is entire and \( f(z) \) is real if and only if \( z \) is real. Explain how you know that \( f \) has at
most one zero.

11.2 Rouche’s Theorem. Suppose \( f \) and \( g \) are analytic on and inside a simple closed contour \( C \). Suppose moreover that \( |f(z)| > |g(z)| \) for all \( z \in C \). Then we shall see that \( f \) and \( f + g \) have the same number of zeros inside \( C \). This result is Rouche’s Theorem. To see why it is so, start by defining the function \( \Psi(t) \) on the interval \( 0 \leq t \leq 1 \):

\[
\Psi(t) = \frac{1}{2\pi i} \int_C \frac{f'(z) + tg'(z)}{f(z) + tg(z)} \, dz.
\]

Observe that this is okay—that is, the denominator of the integrand is never zero:

\[
|f(z) + tg(z)| \geq |f(t)| - t|g(t)| \geq |f(t)| - |g(t)| > 0.
\]

Observe that \( \Psi \) is continuous on the interval \([0, 1]\) and is integer-valued—\( \Psi(t) \) is the number of zeros of \( f + tg \) inside \( C \). Being continuous and integer-valued on the connected set \([0, 1]\), it must be constant. In particular, \( \Psi(0) = \Psi(1) \). This does the job!

\[
\Psi(0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{f(z)} \, dz
\]

is the number of zeros of \( f \) inside \( C \), and

\[
\Psi(1) = \frac{1}{2\pi i} \int_C \frac{f'(z) + g'(z)}{f(z) + g(z)} \, dz
\]

is the number of zeros of \( f + g \) inside \( C \).

Example

How many solutions of the equation \( z^6 - 5z^5 + z^3 - 2 = 0 \) are inside the circle \(|z| = 1|\)? Rouche’s Theorem makes it quite easy to answer this. Simply let \( f(z) = -5z^5 \) and let \( g(z) = z^6 + z^3 - 2 \). Then \( |f(z)| = 5 \) and \( |g(z)| \leq |z|^6 + |z|^3 + 2 = 4 \) for all \(|z| = 1\). Hence \( |f(z)| > |g(z)| \) on the unit circle. From Rouche’s Theorem we know then that \( f \) and \( f + g \) have the same number of zeros inside \(|z| = 1|\). Thus, there are 5 such solutions.

The following nice result follows easily from Rouche’s Theorem. Suppose \( U \) is an open set (i.e., every point of \( U \) is an interior point) and suppose that a sequence \((f_n)\) of functions analytic on \( U \) converges uniformly to the function \( f \). Suppose further that \( f \) is not zero on the circle \( C = \{z : |z - z_0| = R\} \subset U \). Then there is an integer \( N \) so that for all \( n \geq N \), the functions \( f_n \) and \( f \) have the same number of zeros inside \( C \).

This result, called Hurwitz’s Theorem, is an easy consequence of Rouche’s Theorem. Simply
observe that for \( z \in \mathbb{C} \), we have \(|f(z)| > \varepsilon > 0\) for some \( \varepsilon \). Now let \( N \) be large enough to insure that

\[
|f_n(z) - f(z)| < \varepsilon \quad \text{on} \quad C.
\]

It follows from Rouche’s Theorem that \( f \) and \( f + (f_n - f) = f_n \) have the same number of zeros inside \( C \).

**Example**

On any bounded set, the sequence \((f_n)\), where \( f_n(z) = 1 + z + \frac{z^2}{2} + \ldots + \frac{z^n}{n!} \), converges uniformly to

\[
f(z) = e^z, \quad \text{and} \quad f(z) \neq 0 \quad \text{for all} \quad z.
\]

Thus for any \( R \), there is an \( N \) so that for \( n > N \), every zero of \( 1 + z + \frac{z^2}{2} + \ldots + \frac{z^n}{n!} \) has modulus > \( R \). Or to put it another way, given an \( R \) there is an \( N \) so that for \( n > N \) no polynomial \( 1 + z + \frac{z^2}{2} + \ldots + \frac{z^n}{n!} \) has a zero inside the circle of radius \( R \).

**Exercises**

5. How many solutions of \( 3e^z - z = 0 \) are in the disk \(|z| \leq 1| \)? Explain.

6. Show that the polynomial \( z^6 + 4z^2 - 1 \) has exactly two zeros inside the circle \(|z| = 1| \).

7. How many solutions of \( 2z^4 - 2z^3 + 2z^2 - 2z + 9 = 0 \) lie inside the circle \(|z| = 1| \)?

8. Use Rouche’s Theorem to prove that every polynomial of degree \( n \) has exactly \( n \) zeros (counting multiplicity, of course).

9. Let \( C \) be the closed unit disk \(|z| \leq 1| \). Suppose the function \( f \) analytic on \( C \) maps \( C \) into the open unit disk \(|z| < 1| \)—that is, \(|f(z)| < 1 \quad \text{for all} \quad z \in C \). Prove there is exactly one \( w \in C \) such that \( f(w) = w \).

(The point \( w \) is called a **fixed point** of \( f \).)