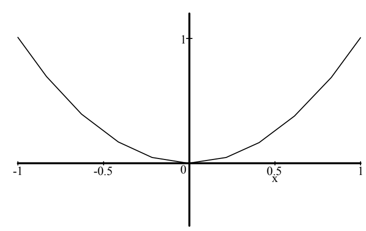
Chapter Two

Complex Functions

2.1. Functions of a real variable. A function $\gamma : I \to \mathbf{C}$ from a set I of reals into the complex numbers $\mathbf{C}$ is actually a familiar concept from elementary calculus. It is simply a function from a subset of the reals into the plane, what we sometimes call a vector-valued function. Assuming the function γ is nice, it provides a vector, or parametric, description of a curve. Thus, the set of all $\{\gamma(t) : \gamma(t) = e^{it} = \cos t + i \sin t = (\cos t, \sin t), 0 \le t \le 2\pi\}$ is the circle of radius one, centered at the origin.

We also already know about the derivatives of such functions. If $\gamma(t) = x(t) + iy(t)$, then the derivative of γ is simply $\gamma'(t) = x'(t) + iy'(t)$, interpreted as a vector in the plane, it is tangent to the curve described by γ at the point $\gamma(t)$.

Example. Let $\gamma(t) = t + it^2$, $-1 \le t \le 1$. One easily sees that this function describes that part of the curve $y = x^2$ between $x = -1$ and $x = 1$:



Another example. Suppose there is a body of mass M ”fixed” at the origin–perhaps the sun–and there is a body of mass m which is free to move–perhaps a planet. Let the location of this second body at time t be given by the complex-valued function $z(t)$. We assume the only force on this mass is the gravitational force of the fixed body. This force f is thus

$$f = \frac{GMm}{|z(t)|^2}\left(-\frac{z(t)}{|z(t)|}\right)$$

where G is the universal gravitational constant. Sir Isaac Newton tells us that

$$mz''(t) = f = \frac{GMm}{|z(t)|^2}\left(-\frac{z(t)}{|z(t)|}\right)$$

Hence,

$$z'' = -\frac{GM}{|z|^3}z$$

Next, let's write this in polar form, $z = re^{i\theta}$:

$$\frac{d^2}{dt^2}(re^{i\theta}) = -\frac{k}{r^2}e^{i\theta}$$

where we have written $GM = k$. Now, let's see what we have.

$$\frac{d}{dt}(re^{i\theta}) = r\frac{d}{dt}(e^{i\theta)}) + \frac{dr}{dt}e^{i\theta}$$

Now,

$$\begin{aligned}\frac{d}{dt}(e^{i\theta)}) &= \frac{d}{dt}(\cos\theta + i\sin\theta)\\ &= (-\sin\theta + i\cos\theta)\frac{d\theta}{dt}\\ &= i(\cos\theta + i\sin\theta)\frac{d\theta}{dt}\\ &= i\frac{d\theta}{dt}e^{i\theta}.\end{aligned}$$

(Additional evidence that our notation $e^{i\theta} = \cos\theta + i\sin\theta$ is reasonable.)
Thus,

$$\begin{aligned}\frac{d}{dt}(re^{i\theta}) &= r\frac{d}{dt}(e^{i\theta)}) + \frac{dr}{dt}e^{i\theta}\\ &= r\left(i\frac{d\theta}{dt}e^{i\theta}\right) + \frac{dr}{dt}e^{i\theta}\\ &= \left(\frac{dr}{dt} + ir\frac{d\theta}{dt}\right)e^{i\theta}.\end{aligned}$$

Now,

$$\frac{d^2}{dt^2}(re^{i\theta}) = \left(\frac{d^2r}{dt^2} + i\frac{dr}{dt}\frac{d\theta}{dt} + ir\frac{d^2\theta}{dt^2}\right)e^{i\theta} + \left(\frac{dr}{dt} + ir\frac{d\theta}{dt}\right)i\frac{d\theta}{dt}e^{i\theta}$$

$$= \left[\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) + i\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\right]e^{i\theta}$$

Now, the equation $\frac{d^2}{dt^2}(re^{i\theta}) = -\frac{k}{r^2}e^{i\theta}$ becomes

$$\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) + i\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right) = -\frac{k}{r^2}.$$

This gives us the two equations

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{k}{r^2},$$

and,

$$r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = 0.$$

Multiply by r and this second equation becomes

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0.$$

This tells us that

$$\alpha = r^2\frac{d\theta}{dt}$$

is a constant. (This constant α is called the **angular momentum**.) This result allows us to get rid of $\frac{d\theta}{dt}$ in the first of the two differential equations above:

$$\frac{d^2r}{dt^2} - r\left(\frac{\alpha}{r^2}\right)^2 = -\frac{k}{r^2}$$

or,

$$\frac{d^2r}{dt^2} - \frac{\alpha^2}{r^3} = -\frac{k}{r^2}.$$

Although this now involves only the one unknown function r, as it stands it is tough to solve. Let's change variables and think of r as a function of θ. Let's also write things in terms of the function $s = \frac{1}{r}$. Then,

$$\frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta} = \frac{\alpha}{r^2}\frac{d}{d\theta}.$$

Hence,

$$\frac{dr}{dt} = \frac{\alpha}{r^2}\frac{dr}{d\theta} = -\alpha\frac{ds}{d\theta},$$

and so

$$\begin{aligned}\frac{d^2r}{dt^2} &= \frac{d}{dt}\left(-\alpha\frac{ds}{d\theta}\right) = \alpha s^2\frac{d}{d\theta}\left(-\alpha\frac{ds}{d\theta}\right)\\ &= -\alpha^2 s^2\frac{d^2s}{d\theta^2},\end{aligned}$$

and our differential equation looks like

$$\frac{d^2r}{dt^2} - \frac{\alpha^2}{r^3} = -\alpha^2 s^2\frac{d^2s}{d\theta^2} - \alpha^2 s^3 = -ks^2,$$

or,

$$\frac{d^2s}{d\theta^2} + s = \frac{k}{\alpha^2}.$$

This one is easy. From high school differential equations class, we remember that

$$s = \frac{1}{r} = A\cos(\theta + \varphi) + \frac{k}{\alpha^2},$$

where A and φ are constants which depend on the initial conditions. At long last,

$$r = \frac{\alpha^2/k}{1 + \varepsilon\cos(\theta + \varphi)},$$

where we have set $\varepsilon = A\alpha^2/k$. The graph of this equation is, of course, a conic section of eccentricity ε.

Exercises

1. a)What curve is described by the function $\gamma(t) = (3t+4) + i(t-6),\ 0 \le t \le 1$?
b)Suppose z and w are complex numbers. What is the curve described by $\gamma(t) = (1-t)w + tz,\ 0 \le t \le 1$?

2. Find a function γ that describes that part of the curve $y = 4x^3 + 1$ between $x = 0$ and $x = 10$.

3. Find a function γ that describes the circle of radius 2 centered at $z = 3 - 2i$.

4. Note that in the discussion of the motion of a body in a central gravitational force field, it was assumed that the angular momentum α is nonzero. Explain what happens in case $\alpha = 0$.

2.2 Functions of a complex variable. The real excitement begins when we consider function $f : D \to \mathbf{C}$ in which the domain D is a subset of the complex numbers. In some sense, these too are familiar to us from elementary calculus—they are simply functions from a subset of the plane into the plane:

$$f(z) = f(x,y) = u(x,y) + iv(x,y) = (u(x,y), v(x,y))$$

Thus $f(z) = z^2$ looks like $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2xyi$. In other words, $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. The complex perspective, as we shall see, generally provides richer and more profitable insights into these functions.

The definition of the **limit** of a function f at a point $z = z_0$ is essentially the same as that which we learned in elementary calculus:

$$\lim_{z \to z_0} f(z) = L$$

means that given an $\varepsilon > 0$, there is a δ so that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$. As you could guess, we say that f is **continuous** at z_0 if it is true that $\lim_{z \to z_0} f(z) = f(z_0)$. If f is continuous at each point of its domain, we say simply that f is **continuous.**

Suppose both $\lim_{z \to z_0} f(z)$ and $\lim_{z \to z_0} g(z)$ exist. Then the following properties are easy to establish:

$$\lim_{z\to z_0}[f(z)\pm g(z)] = \lim_{z\to z_0} f(z) \pm \lim_{z\to z_0} g(z)$$

$$\lim_{z\to z_0} f(z)g(z) = \lim_{z\to z_0} f(z) \lim_{z\to z_0} g(z)$$

and,

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)}$$

provided, of course, that $\lim_{z\to z_0} g(z) \neq 0$.

It now follows at once from these properties that the sum, difference, product, and quotient of two functions continuous at z_0 are also continuous at z_0. (We must, as usual, except the dreaded 0 in the denominator.)

It should not be too difficult to convince yourself that if $z = (x,y)$, $z_0 = (x_0,y_0)$, and $f(z) = u(x,y) + iv(x,y)$, then

$$\lim_{z\to z_0} f(z) = \lim_{(x,y)\to(x_0,y_0)} u(x,y) + i \lim_{(x,y)\to(x_0,y_0)} v(x,y)$$

Thus f is continuous at $z_0 = (x_0,y_0)$ precisely when u and v are.

Our next step is the definition of the derivative of a complex function f. It is the obvious thing. Suppose f is a function and z_0 is an interior point of the domain of f. The **derivative** $f'(z_0)$ of f is

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Example

Suppose $f(z) = z^2$. Then, letting $\Delta z = z - z_0$, we have

$$
\begin{aligned}
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{2z_0\Delta z + (\Delta z)^2}{\Delta z} \\
&= \lim_{\Delta z \to 0} (2z_0 + \Delta z) \\
&= 2z_0
\end{aligned}
$$

No surprise here–the function $f(z) = z^2$ has a derivative at every z, and it's simply $2z$.

Another Example

Let $f(z) = z\overline{z}$. Then,

$$
\begin{aligned}
\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)\overline{(z_0 + \Delta z)} - z_0\overline{z}_0}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{z_0\overline{\Delta z} + \overline{z}_0\Delta z + \Delta z\overline{\Delta z}}{\Delta z} \\
&= \lim_{\Delta z \to 0} \left(\overline{z}_0 + \overline{\Delta z} + z_0 \frac{\overline{\Delta z}}{\Delta z} \right)
\end{aligned}
$$

Suppose this limit exists, and choose $\Delta z = (\Delta x, 0)$. Then,

$$
\begin{aligned}
\lim_{\Delta z \to 0} \left(\overline{z}_0 + \overline{\Delta z} + z_0 \frac{\overline{\Delta z}}{\Delta z} \right) &= \lim_{\Delta x \to 0} \left(\overline{z}_0 + \Delta x + z_0 \frac{\Delta x}{\Delta x} \right) \\
&= \overline{z}_0 + z_0
\end{aligned}
$$

Now, choose $\Delta z = (0, \Delta y)$. Then,

$$
\begin{aligned}
\lim_{\Delta z \to 0} \left(\overline{z}_0 + \overline{\Delta z} + z_0 \frac{\overline{\Delta z}}{\Delta z} \right) &= \lim_{\Delta y \to 0} \left(\overline{z}_0 - i\Delta y - z_0 \frac{i\Delta y}{i\Delta y} \right) \\
&= \overline{z}_0 - z_0
\end{aligned}
$$

Thus, we must have $\overline{z}_0 + z_0 = \overline{z}_0 - z_0$, or $z_0 = 0$. In other words, there is no chance of this limit's existing, except possibly at $z_0 = 0$. So, this function does not have a derivative at most places.

Now, take another look at the first of these two examples. It looks exactly like what you did in Mrs. Turner's 3^{rd} grade calculus class for plain old real-valued functions. Meditate on this and you will be convinced that all the "usual" results for real-valued functions also hold for these new complex functions: the derivative of a constant is zero, the derivative of the sum of two functions is the sum of the derivatives, the "product" and "quotient" rules for derivatives are valid, the chain rule for the composition of functions holds, *etc., etc.* For proofs, you need only go back to your elementary calculus book and change x's to z's.

A bit of jargon is in order. If f has a derivative at z_0, we say that f is **differentiable** at z_0. If f is differentiable at every point of a neighborhood of z_0, we say that f is **analytic** at z_0. (A set S is a **neighborhood** of z_0 if there is a disk $D = \{z : |z - z_0| < r, r > 0\}$ so that $D \subset S$.) If f is analytic at every point of some set S, we say that f is **analytic on** S. A function that is analytic on the set of all complex numbers is said to be an **entire** function.

Exercises

5. Suppose $f(z) = 3xy + i(x - y^2)$. Find $\lim\limits_{z \to 3+2i} f(z)$, or explain carefully why it does not exist.

6. Prove that if f has a derivative at z, then f is continuous at z.

7. Find all points at which the valued function f defined by $f(z) = \overline{z}$ has a derivative.

8. Find all points at which the valued function f defined by

$$f(z) = (2 + i)z^3 - iz^2 + 4z - (1 + 7i)$$

has a derivative.

9. Is the function f given by

$$f(z) = \begin{cases} \frac{(\overline{z})^2}{z} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$$

differentiable at $z = 0$? Explain.

2.3. Derivatives. Suppose the function f given by $f(z) = u(x,y) + iv(x,y)$ has a derivative at $z = z_0 = (x_0, y_0)$. We know this means there is a number $f'(z_0)$ so that

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

Choose $\Delta z = (\Delta x, 0) = \Delta x$. Then,

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\
&= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \right] \\
&= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0)
\end{aligned}$$

Next, choose $\Delta z = (0, \Delta y) = i\Delta y$. Then,

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\
&= \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \left[\frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} - i \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y} \right] \\
&= \frac{\partial v}{\partial y}(x_0, y_0) - i \frac{\partial u}{\partial y}(x_0, y_0)
\end{aligned}$$

We have two different expressions for the derivative $f'(z_0)$, and so

$$\frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i \frac{\partial u}{\partial y}(x_0, y_0)$$

or,

$$\begin{aligned}
\frac{\partial u}{\partial x}(x_0, y_0) &= \frac{\partial v}{\partial y}(x_0, y_0), \\
\frac{\partial u}{\partial y}(x_0, y_0) &= -\frac{\partial v}{\partial x}(x_0, y_0)
\end{aligned}$$

These equations are called the **Cauchy-Riemann Equations**.

We have shown that if f has a derivative at a point z_0, then its real and imaginary parts satisfy these equations. Even more exciting is the fact that if the real and imaginary parts of f satisfy these equations and if in addition, they have continuous first partial derivatives, then the function f has a derivative. Specifically, suppose $u(x, y)$ and $v(x, y)$ have partial derivatives in a neighborhood of $z_0 = (x_0, y_0)$, suppose these derivatives are continuous at z_0, and suppose

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0),$$
$$\frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0).$$

We shall see that f is differentiable at z_0.

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$
$$= \frac{[u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0)]+i[v(x_0+\Delta x,y_0+\Delta y)-v(x_0,y_0)]}{\Delta x+i\Delta y}.$$

Observe that

$$u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0) = [u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0+\Delta y)] + [u(x_0,y_0+\Delta y)-u(x_0,y_0)].$$

Thus,

$$u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0+\Delta y) = \Delta x\frac{\partial u}{\partial x}(\xi,y_0+\Delta y),$$

and,

$$\frac{\partial u}{\partial x}(\xi,y_0+\Delta y) = \frac{\partial u}{\partial x}(x_0,y_0)+\varepsilon_1,$$

where,

$$\lim_{\Delta z\to 0}\varepsilon_1 = 0.$$

Thus,

$$u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0+\Delta y) = \Delta x\left[\frac{\partial u}{\partial x}(x_0,y_0)+\varepsilon_1\right].$$

Proceeding similarly, we get

$$\begin{aligned}
&\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} \\
&= \frac{[u(x_0+\Delta x, y_0+\Delta y)-u(x_0,y_0)]+i[v(x_0+\Delta x, y_0+\Delta y)-v(x_0,y_0)]}{\Delta x+i\Delta y} \\
&= \frac{\Delta x\left[\frac{\partial u}{\partial x}(x_0,y_0)+\varepsilon_1+i\frac{\partial v}{\partial x}(x_0,y_0)+i\varepsilon_2\right]+\Delta y\left[\frac{\partial u}{\partial y}(x_0,y_0)+\varepsilon_3+i\frac{\partial v}{\partial y}(x_0,y_0)+i\varepsilon_4\right]}{\Delta x+i\Delta y},.
\end{aligned}$$

where $\varepsilon_i \to 0$ as $\Delta z \to 0$. Now, unleash the Cauchy-Riemann equations on this quotient and obtain,

$$\begin{aligned}
&\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} \\
&= \frac{\Delta x\left[\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right]+i\Delta y\left[\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right]}{\Delta x+i\Delta y}+\frac{\text{stuff}}{\Delta x+i\Delta y} \\
&= \left[\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right]+\frac{\text{stuff}}{\Delta x+i\Delta y}.
\end{aligned}$$

Here,

$$\text{stuff} = \Delta x(\varepsilon_1+i\varepsilon_2)+\Delta y(\varepsilon_3+i\varepsilon_4).$$

It's easy to show that

$$\lim_{\Delta z\to 0}\frac{\text{stuff}}{\Delta z} = 0,$$

and so,

$$\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}.$$

In particular we have, as promised, shown that f is differentiable at z_0.

Example

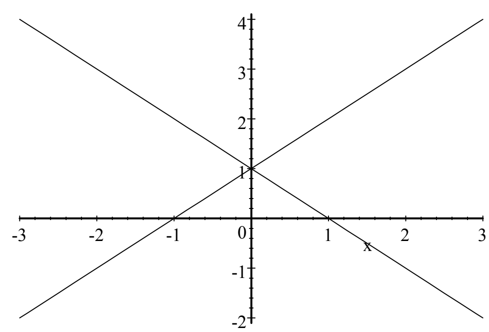
Let's find all points at which the function f given by $f(z) = x^3 - i(1-y)^3$ is differentiable. Here we have $u = x^3$ and $v = -(1-y)^3$. The Cauchy-Riemann equations thus look like

$$\begin{aligned}
3x^2 &= 3(1-y)^2, \text{ and} \\
0 &= 0.
\end{aligned}$$

The partial derivatives of u and v are nice and continuous everywhere, so f will be differentiable everywhere the C-R equations are satisfied. That is, everywhere

$$x^2 = (1-y)^2; \text{ that is, where}$$
$$x = 1-y, \text{ or } x = -1+y.$$

This is simply the set of all points on the cross formed by the two straight lines



Exercises

10. At what points is the function f given by $f(z) = x^3 + i(1-y)^3$ analytic? Explain.

11. Do the real and imaginary parts of the function f in Exercise **9** satisfy the Cauchy-Riemann equations at $z = 0$? What do you make of your answer?

12. Find all points at which $f(z) = 2y - ix$ is differentiable.

13. Suppose f is analytic on a connected open set D, and $f'(z) = 0$ for all $z \epsilon D$. Prove that f is constant.

14. Find all points at which

$$f(z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

is differentiable. At what points is f analytic? Explain.

15. Suppose f is analytic on the set D, and suppose Ref is constant on D. Is f necessarily constant on D? Explain.

16. Suppose f is analytic on the set D, and suppose $|f(z)|$ is constant on D. Is f necessarily constant on D? Explain.