Chapter Four
Integration

4.1. Introduction. If $\gamma : D \to \mathbb{C}$ is simply a function on a real interval $D = [\alpha, \beta]$, then the integral $\int_{\alpha}^{\beta} \gamma(t)dt$ is, of course, simply an ordered pair of everyday 3\textsuperscript{rd} grade calculus integrals:

$$\int_{\alpha}^{\beta} \gamma(t)dt = \int_{\alpha}^{\beta} x(t)dt + i \int_{\alpha}^{\beta} y(t)dt,$$

where $\gamma(t) = x(t) + iy(t)$. Thus, for example,

$$\int_{0}^{1} [(t^2 + 1) + it^3]dt = \frac{4}{3} + \frac{i}{4}.$$

Nothing really new here. The excitement begins when we consider the idea of an integral of an honest-to-goodness complex function $f : D \to \mathbb{C}$, where $D$ is a subset of the complex plane. Let’s define the integral of such things; it is pretty much a straight-forward extension to two dimensions of what we did in one dimension back in Mrs. Turner’s class.

Suppose $f$ is a complex-valued function on a subset of the complex plane and suppose $a$ and $b$ are complex numbers in the domain of $f$. In one dimension, there is just one way to get from one number to the other; here we must also specify a path from $a$ to $b$. Let $C$ be a path from $a$ to $b$, and we must also require that $C$ be a subset of the domain of $f$. 

\[\text{Diagram of path from } a \text{ to } b\]
Note we do not even require that \(a \neq b\); but in case \(a = b\), we must specify an orientation for the closed path \(C\). We call a path, or curve, **closed** in case the initial and terminal points are the same, and a **simple closed** path is one in which no other points coincide. Next, let \(P\) be a **partition** of the curve; that is, \(P = \{z_0, z_1, z_2, \ldots, z_n\}\) is a finite subset of \(C\), such that \(a = z_0, b = z_n\), and such that \(z_j\) comes immediately after \(z_{j-1}\) as we travel along \(C\) from \(a\) to \(b\).

A Riemann sum associated with the partition \(P\) is just what it is in the real case:

\[
S(P) = \sum_{j=1}^{n} f(z^*_j) \Delta z_j,
\]

where \(z^*_j\) is a point on the arc between \(z_{j-1}\) and \(z_j\), and \(\Delta z_j = z_j - z_{j-1}\). (Note that for a given partition \(P\), there are many \(S(P)\)—depending on how the points \(z^*_j\) are chosen.) If there is a number \(L\) so that given any \(\varepsilon > 0\), there is a partition \(P_\varepsilon\) of \(C\) such that

\[
|S(P) - L| < \varepsilon
\]

whenever \(P \supset P_\varepsilon\), then \(f\) is said to be integrable on \(C\) and the number \(L\) is called the **integral of** \(f\) on \(C\). This number \(L\) is usually written \(\int_C f(z) dz\).

Some properties of integrals are more or less evident from looking at Riemann sums:

\[
\int_C cf(z) dz = c \int_C f(z) dz
\]

for any complex constant \(c\).
\[ \int_C (f(z) + g(z))dz = \int_C f(z)dz + \int_C g(z)dz \]

4.2 Evaluating integrals. Now, how on Earth do we ever find such an integral? Let \( \gamma : [\alpha, \beta] \rightarrow C \) be a complex description of the curve \( C \). We partition \( C \) by partitioning the interval \([\alpha, \beta]\) in the usual way: \( \alpha = t_0 < t_1 < t_2 < \ldots < t_n = \beta \). Then \( \{a = \gamma(\alpha), \gamma(t_1), \gamma(t_2), \ldots, \gamma(\beta) = b\} \) is partition of \( C \). (Recall we assume that \( \gamma'(t) \neq 0 \) for a complex description of a curve \( C \).) A corresponding Riemann sum looks like

\[ S(P) = \sum_{j=1}^{n} f(\gamma(t_j^*)) (\gamma(t_j) - \gamma(t_{j-1})). \]

We have chosen the points \( z_j^* = \gamma(t_j^*) \), where \( t_{j-1} \leq t_j^* \leq t_j \). Next, multiply each term in the sum by 1 in disguise:

\[ S(P) = \sum_{j=1}^{n} f(\gamma(t_j^*)) (\frac{\gamma(t_j) - \gamma(t_{j-1})}{t_j - t_{j-1}})(t_j - t_{j-1}). \]

I hope it is now reasonably convincing that ”in the limit”, we have

\[ \int_C f(z)dz = \int_{\alpha}^{\beta} f(\gamma(t))\gamma'(t)dt. \]

(We are, of course, assuming that the derivative \( \gamma' \) exists.)

Example

We shall find the integral of \( f(z) = (x^2 + y) + i(xy) \) from \( a = 0 \) to \( b = 1 + i \) along three different paths, or contours, as some call them.

First, let \( C_1 \) be the part of the parabola \( y = x^2 \) connecting the two points. A complex description of \( C_1 \) is \( \gamma_1(t) = t + it^2, \ 0 \leq t \leq 1 \):
Now, $\gamma_1'(t) = 1 + 2ti$, and $f(\gamma_1(t)) = (t^2 + t^2) + it^2 = 2t^2 + it^3$. Hence,

$$\int_{C_1} f(z)dz = \int_0^1 f(\gamma_1(t))\gamma_1'(t)dt = \int_0^1 (2t^2 + it^3)(1 + 2ti)dt = \int_0^1 (2t^2 - 2t^4 + 5t^3i)dt = \frac{4}{15} + \frac{5}{4}i$$

Next, let’s integrate along the straight line segment $C_2$ joining 0 and $1 + i$.

Here we have $\gamma_2(t) = t + it$, $0 \leq t \leq 1$. Thus, $\gamma_2'(t) = 1 + i$, and our integral looks like
\[
\int_{C_3} f(z) \, dz = \int_{C_{31}} f(z) \, dz + \int_{C_{32}} f(z) \, dz.
\]

Finally, let’s integrate along \( C_3 \), the path consisting of the line segment from 0 to 1 together with the segment from 1 to 1 + \( i \).

We shall do this in two parts: \( C_{31} \), the line from 0 to 1; and \( C_{32} \), the line from 1 to 1 + \( i \). Then we have

\[
\int_{C_3} f(z) \, dz = \int_{C_{31}} f(z) \, dz + \int_{C_{32}} f(z) \, dz.
\]

For \( C_{31} \) we have \( \gamma(t) = t, 0 \leq t \leq 1 \). Hence,

\[
\int_{C_{31}} f(z) \, dz = \int_0^1 t^2 \, dt = \frac{1}{3}.
\]

For \( C_{32} \) we have \( \gamma(t) = 1 + it, 0 \leq t \leq 1 \). Hence,

\[
\int_{C_{32}} f(z) \, dz = \int_0^1 (1 + it) \, dt = -\frac{1}{2} + \frac{3}{2}i.
\]
Thus,

\[ \int_{C_1} f(z)\,dz = \int_{C_{31}} f(z)\,dz + \int_{C_{32}} f(z)\,dz \]
\[ = -\frac{1}{6} + \frac{3}{2}i. \]

Suppose there is a number \( M \) so that \(|f(z)| \leq M\) for all \( z \in C \). Then

\[ \left| \int_{C} f(z)\,dz \right| = \left| \int_{a}^{\beta} f(\gamma(t))\gamma'(t)\,dt \right| \]
\[ \leq \int_{a}^{\beta} |f(\gamma(t))\gamma'(t)|\,dt \]
\[ \leq M \int_{a}^{\beta} |\gamma'(t)|\,dt = ML, \]

where \( L = \int_{a}^{\beta} |\gamma'(t)|\,dt \) is the length of \( C \).

**Exercises**

1. Evaluate the integral \( \int_{C} z\,dz \), where \( C \) is the parabola \( y = x^2 \) from 0 to 1 + i.

2. Evaluate \( \int_{C} \frac{1}{z}\,dz \), where \( C \) is the circle of radius 2 centered at 0 oriented counterclockwise.

4. Evaluate \( \int_{C} f(z)\,dz \), where \( C \) is the curve \( y = x^3 \) from \(-1 - i\) to 1 + i, and

\[ f(z) = \begin{cases} 
1 & \text{for } y < 0 \\
4y & \text{for } y \geq 0 
\end{cases} \]

5. Let \( C \) be the part of the circle \( \gamma(t) = e^{it} \) in the first quadrant from \( a = 1 \) to \( b = i \). Find as small an upper bound as you can for \( \left| \int_{C} (z^2 - \overline{z}^4 + 5)\,dz \right| \).
6. Evaluate \( \int_C f(z) \, dz \) where \( f(z) = z + 2 \bar{z} \) and \( C \) is the path from \( z = 0 \) to \( z = 1 + 2i \) consisting of the line segment from 0 to 1 together with the segment from 1 to \( 1 + 2i \).

4.3 Antiderivatives. Suppose \( D \) is a subset of the reals and \( \gamma : D \to \mathbb{C} \) is differentiable at \( t \). Suppose further that \( g \) is differentiable at \( \gamma(t) \). Then let’s see about the derivative of the composition \( g(\gamma(t)) \). It is, in fact, exactly what one would guess. First,

\[
g(\gamma(t)) = u(x(t), y(t)) + iv(x(t), y(t)),
\]

where \( g(z) = u(x, y) + iv(x, y) \) and \( \gamma(t) = x(t) + iy(t) \). Then,

\[
\frac{d}{dt} g(\gamma(t)) = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + i \left( \frac{\partial v}{\partial x} \frac{dx}{dt} + \frac{\partial v}{\partial y} \frac{dy}{dt} \right).
\]

The places at which the functions on the right-hand side of the equation are evaluated are obvious. Now, apply the Cauchy-Riemann equations:

\[
\frac{d}{dt} g(\gamma(t)) = \frac{\partial u}{\partial x} \frac{dx}{dt} - \frac{\partial v}{\partial x} \frac{dy}{dt} + i \left( \frac{\partial v}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} \right)
\]

\[
= \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \left( \frac{dx}{dt} + i \frac{dy}{dt} \right)
\]

\[
= g'(\gamma(t)) \gamma'(t).
\]

The nicest result in the world!

Now, back to integrals. Let \( F : D \to \mathbb{C} \) and suppose \( F'(z) = f(z) \) in \( D \). Suppose moreover that \( a \) and \( b \) are in \( D \) and that \( C \subset D \) is a contour from \( a \) to \( b \). Then

\[
\int_C f(z) \, dz = \int_a^b f(\gamma(t)) \gamma'(t) \, dt,
\]

where \( \gamma : [\alpha, \beta] \to C \) describes \( C \). From our introductory discussion, we know that

\[
\frac{d}{dt} F(\gamma(t)) = F'(\gamma(t)) \gamma'(t) = f(\gamma(t)) \gamma'(t).
\]

Hence,
This is very pleasing. Note that integral depends only on the points $a$ and $b$ and not at all on the path $C$. We say the integral is \textbf{path independent}. Observe that this is equivalent to saying that the integral of $f$ around any closed path is 0. We have thus shown that if in $D$ the integrand $f$ is the derivative of a function $F$, then any integral $\int_C f(z)dz$ for $C \subset D$ is path independent.

\textbf{Example}

Let $C$ be the curve $y = \frac{1}{x^3}$ from the point $z = 1 + i$ to the point $z = 3 + \frac{i}{9}$. Let’s find

$$\int_C z^2 dz.$$

This is easy—we know that $F'(z) = z^2$, where $F(z) = \frac{1}{3}z^3$. Thus,

$$\int_C z^2 dz = \frac{1}{3} \left[ (1 + i)^3 - \left(3 + \frac{i}{9}\right)^3 \right]$$

$$= -\frac{260}{27} - \frac{728}{2187}i$$

Now, instead of assuming $f$ has an antiderivative, let us suppose that the integral of $f$ between any two points in the domain is independent of path and that $f$ is continuous. Assume also that every point in the domain $D$ is an interior point of $D$ and that $D$ is connected. We shall see that in this case, $f$ has an antiderivative. To do so, let $z_0$ be any point in $D$, and define the function $F$ by

$$F(z) = \int_{C_z} f(z)dz,$$

where $C_z$ is any path in $D$ from $z_0$ to $z$. Here is important that the integral is path independent, otherwise $F(z)$ would not be well-defined. Note also we need the assumption that $D$ is connected in order to be sure there always is at least one such path.
Now, for the computation of the derivative of $F$:

$$F(z + \Delta z) - F(z) = \int_{L_{\Delta z}} f(s) \, ds,$$

where $L_{\Delta z}$ is the line segment from $z$ to $z + \Delta z$.

Next, observe that $\int_{L_{\Delta z}} ds = \Delta z$. Thus, $f(z) = \frac{1}{\Delta z} \int_{L_{\Delta z}} f(z) \, ds$, and we have

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z} \int_{L_{\Delta z}} (f(s) - f(z)) \, ds.$$

Now then,

$$\left| \frac{1}{\Delta z} \int_{L_{\Delta z}} (f(s) - f(z)) \, ds \right| \leq \left| \frac{1}{\Delta z} \right| |\Delta z| \max \{ |f(s) - f(z)| : s \in L_{\Delta z} \}$$

$$\leq \max \{ |f(s) - f(z)| : s \in L_{\Delta z} \}.$$

We know $f$ is continuous at $z$, and so $\lim_{\Delta z \to 0} \max \{ |f(s) - f(z)| : s \in L_{\Delta z} \} = 0$. Hence,

$$\lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) = \lim_{\Delta z \to 0} \left( \frac{1}{\Delta z} \int_{L_{\Delta z}} (f(s) - f(z)) \, ds \right)$$

$$= 0.$$
In other words, \( F'(z) = f(z) \), and so, just as promised, \( f \) has an antiderivative! Let’s summarize what we have shown in this section:

Suppose \( f : D \to \mathbb{C} \) is continuous, where \( D \) is connected and every point of \( D \) is an interior point. Then \( f \) has an antiderivative if and only if the integral between any two points of \( D \) is path independent.

**Exercises**

7. Suppose \( C \) is any curve from 0 to \( \pi + 2i \). Evaluate the integral

\[
\int_{C} \cos \left( \frac{z}{2} \right) dz.
\]

8. a) Let \( F(z) = \log z, -\frac{3}{4} \pi < \arg z < \frac{5}{4} \pi \). Show that the derivative \( F'(z) = \frac{1}{z} \).
   b) Let \( G(z) = \log z, -\frac{5}{4} \pi < \arg z < \frac{7}{4} \pi \). Show that the derivative \( G'(z) = \frac{1}{z} \).
   c) Let \( C_1 \) be a curve in the right-half plane \( D_1 = \{ z : \Re z \geq 0 \} \) from \(-i\) to \( i \) that does not pass through the origin. Find the integral

\[
\int_{C_1} \frac{1}{z} dz.
\]

d) Let \( C_2 \) be a curve in the left-half plane \( D_2 = \{ z : \Re z \leq 0 \} \) from \(-i\) to \( i \) that does not pass through the origin. Find the integral

\[
\int_{C_2} \frac{1}{z} dz.
\]

9. Let \( C \) be the circle of radius 1 centered at 0 with the *clockwise* orientation. Find

\[
\int_{C} \frac{1}{z} dz.
\]

10. a) Let \( H(z) = z^c, -\pi < \arg z < \pi \). Find the derivative \( H'(z) \).
    b) Let \( K(z) = z^c, -\frac{\pi}{4} < \arg z < \frac{7\pi}{4} \). Find the derivative \( K'(z) \).
    c) Let \( C \) be any path from \(-1\) to \( 1 \) that lies completely in the upper half-plane and does not pass through the origin. (Upper half-plane = \( \{ z : \Im z \geq 0 \} \).) Find

4.10
\[ \int_C F(z)dz, \]

where \( F(z) = z^i, -\pi < \arg z \leq \pi. \)

11. Suppose \( P \) is a polynomial and \( C \) is a closed curve. Explain how you know that \( \int_C P(z)dz = 0. \)