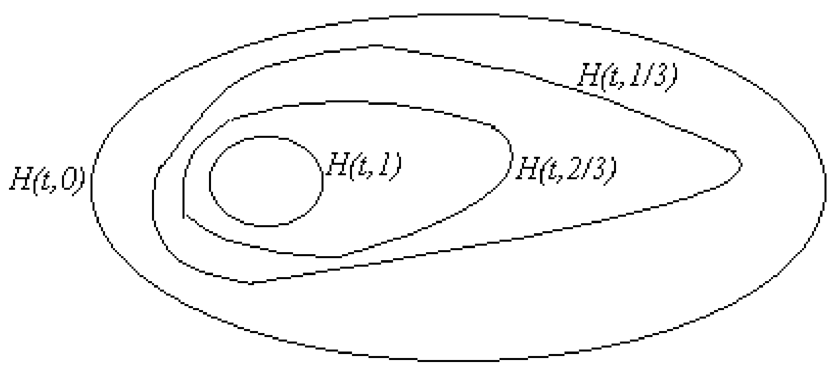
Chapter Five

Cauchy's Theorem

5.1. Homotopy. Suppose D is a connected subset of the plane such that every point of D is an interior point—we call such a set a **region**—and let C_1 and C_2 be oriented closed curves in D. We say C_1 is **homotopic** to C_2 in D if there is a continuous function $H : S \to D$, where S is the square $S = \{(t,s) : 0 \le s,t \le 1\}$, such that $H(t,0)$ describes C_1 and $H(t,1)$ describes C_2, and for each fixed s, the function $H(t,s)$ describes a closed curve C_s in D. The function H is called a **homotopy** between C_1 and C_2. Note that if C_1 is homotopic to C_2 in D, then C_2 is homotopic to C_1 in D. Just observe that the function $K(t,s) = H(t,1-s)$ is a homotopy.

It is convenient to consider a point to be a closed curve. The point c is a described by a constant function $\gamma(t) = c$. We thus speak of a closed curve C being homotopic to a constant—we sometimes say C is **contractible** to a point.

Emotionally, the fact that two closed curves are homotopic in D means that one can be continuously deformed into the other in D.

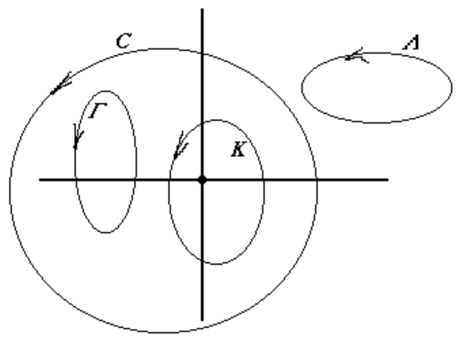


Example

Let D be the annular region $D = \{z : 1 < |z| < 5\}$. Suppose C_1 is the circle described by $\gamma_1(t) = 2e^{i2\pi t}$, $0 \le t \le 1$; and C_2 is the circle described by $\gamma_2(t) = 4e^{i2\pi t}$, $0 \le t \le 1$. Then $H(t,s) = (2+2s)e^{i2\pi t}$ is a homotopy in D between C_1 and C_2. Suppose C_3 is the same circle as C_2 but with the opposite orientation; that is, a description is given by $\gamma_3(t) = 4e^{-i2\pi t}$, $0 \le t \le 1$. A homotopy between C_1 and C_3 is not too easy to construct—in fact, it is not possible! The moral: orientation counts. From now on, the term "closed curve" will mean an oriented closed curve.

Another Example

Let D be the set obtained by removing the point $z = 0$ from the plane. Take a look at the picture. Meditate on it and convince yourself that C and K are homotopic in D, but Γ and Λ are homotopic in D, while K and Γ are not homotopic in D.



Exercises

1. Suppose C_1 is homotopic to C_2 in D, and C_2 is homotopic to C_3 in D. Prove that C_1 is homotopic to C_3 in D.

2. Explain how you know that any two closed curves in the plane **C** are homotopic in **C**.

3. A region D is said to be **simply connected** if every closed curve in D is contractible to a point in D. Prove that any two closed curves in a simply connected region are homotopic in D.

5.2 Cauchy's Theorem. Suppose C_1 and C_2 are closed curves in a region D that are homotopic in D, and suppose f is a function analytic on D. Let $H(t,s)$ be a homotopy between C_1 and C_2. For each s, the function $\gamma_s(t)$ describes a closed curve C_s in D. Let $I(s)$ be given by

$$I(s) = \int_{C_s} f(z)dz.$$

Then,

$$I(s) = \int_0^1 f(H(t,s)) \frac{\partial H(t,s)}{\partial t} dt.$$

Now let's look at the derivative of $I(s)$. We assume everything is nice enough to allow us to differentiate under the integral:

$$\begin{aligned}
I'(s) &= \frac{d}{ds}\left[\int_0^1 f(H(t,s)) \frac{\partial H(t,s)}{\partial t} dt \right] \\
&= \int_0^1 \left[f'(H(t,s)) \frac{\partial H(t,s)}{\partial s} \frac{\partial H(t,s)}{\partial t} + f(H(t,s)) \frac{\partial^2 H(t,s)}{\partial s \partial t} \right] dt \\
&= \int_0^1 \left[f'(H(t,s)) \frac{\partial H(t,s)}{\partial s} \frac{\partial H(t,s)}{\partial t} + f(H(t,s)) \frac{\partial^2 H(t,s)}{\partial t \partial s} \right] dt \\
&= \int_0^1 \frac{\partial}{\partial t}\left[f(H(t,s)) \frac{\partial H(t,s)}{\partial s} \right] dt \\
&= f(H(1,s)) \frac{\partial H(1,s)}{\partial s} - f(H(0,s)) \frac{\partial H(0,s)}{\partial s}.
\end{aligned}$$

But we know each $H(t,s)$ describes a closed curve, and so $H(0,s) = H(1,s)$ for all s. Thus,

$$I'(s) = f(H(1,s)) \frac{\partial H(1,s)}{\partial s} - f(H(0,s)) \frac{\partial H(0,s)}{\partial s} = 0,$$

which means $I(s)$ is constant! In particular, $I(0) = I(1)$, or

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

This is a big deal. We have shown that if C_1 and C_2 are closed curves in a region D that are homotopic in D, and f is analytic on D, then $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.

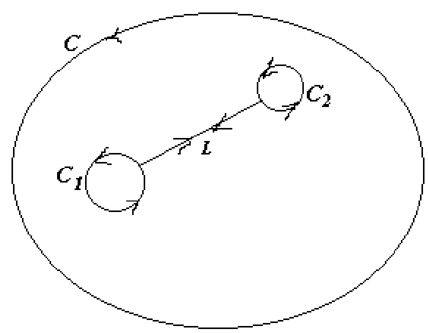
An easy corollary of this result is the celebrated **Cauchy's Theorem**, which says that if f is analytic on a simply connected region D, then for any closed curve C in D,

$$\int_C f(z)dz = 0.$$

In court testimony, one is admonished to tell the truth, the whole truth, and nothing but the truth. Well, so far in this chapter, we have told the truth and nothing but the truth, but we have not quite told the whole truth. We assumed all sorts of continuous derivatives in the preceding discussion. These are not always necessary—specifically, the results can be proved true without all our smoothness assumptions—think about approximation.

Example

Look at the picture below and convince your self that the path C is homotopic to the closed path consisting of the two curves C_1 and C_2 together with the line L. We traverse the line twice, once from C_1 to C_2 and once from C_2 to C_1.



Observe then that an integral over this closed path is simply the sum of the integrals over C_1 and C_2, since the two integrals along L , being in opposite directions, would sum to zero. Thus, if f is analytic in the region bounded by these curves (the region with two holes in it), then we know that

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz.$$

Exercises

4. Prove Cauchy's Theorem.

5. Let S be the square with sides $x = \pm 100$, and $y = \pm 100$ with the counterclockwise orientation. Find

$$\int_S \frac{1}{z}\,dz.$$

6. a)Find $\int_C \frac{1}{z-1}\,dz$, where C is any circle centered at $z = 1$ with the usual counterclockwise orientation: $\gamma(t) = 1 + Ae^{2\pi it}$, $0 \le t \le 1$.

b)Find $\int_C \frac{1}{z+1}\,dz$, where C is any circle centered at $z = -1$ with the usual counterclockwise orientation.

c)Find $\int_C \frac{1}{z^2-1}\,dz$, where C is the ellipse $4x^2 + y^2 = 100$ with the counterclockwise orientation. [Hint: partial fractions]

d)Find $\int_C \frac{1}{z^2-1}\,dz$, where C is the circle $x^2 - 10x + y^2 = 0$ with the counterclockwise orientation.

8. Evaluate $\int_C \mathrm{Log}\,(z+3)\,dz$, where C is the circle $|z| = 2$ oriented counterclockwise.

9. Evaluate $\int_C \frac{1}{z^n}\,dz$ where C is the circle described by $\gamma(t) = e^{2\pi it}$, $0 \le t \le 1$, and n is an integer $\neq 1$.

10. a)Does the function $f(z) = \frac{1}{z}$ have an antiderivative on the set of all $z \neq 0$? Explain.

b)How about $f(z) = \frac{1}{z^n}$, n an integer $\neq 1$?

11. Find as large a set D as you can so that the function $f(z) = e^{z^2}$ have an antiderivative on D.

12. Explain how you know that every function analytic in a simply connected (cf. Exercise 3) region D is the derivative of a function analytic in D.