Chapter Seven

Harmonic Functions

7.1. The Laplace equation. The Fourier law of heat conduction says that the rate at which heat passes across a surface S is proportional to the flux, or surface integral, of the temperature gradient on the surface:

$$k \iint_S \nabla T \cdot dA.$$

Here k is the constant of proportionality, generally called the *thermal conductivity* of the substance (We assume uniform stuff.). We further assume no heat sources or sinks, and we assume steady-state conditions—the temperature does not depend on time. Now if we take S to be an arbitrary closed surface, then this rate of flow must be 0:

$$k \iint_S \nabla T \cdot dA = 0.$$

Otherwise there would be more heat entering the region B bounded by S than is coming out, or vice-versa. Now, apply the celebrated Divergence Theorem to conclude that

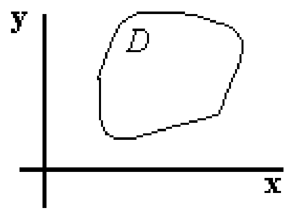
$$\iiint_B (\nabla \cdot \nabla T) dV = 0,$$

where B is the region bounded by the closed surface S. But since the region B is completely arbitrary, this means that

$$\nabla \cdot \nabla T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0.$$

This is the world-famous **Laplace Equation**.

Now consider a slab of heat conducting material,



in which we assume there is no heat flow in the z-direction. Equivalently, we could assume we are looking at the cross-section of a long rod in which there is no longitudinal heat flow. In other words, we are looking at a two-dimensional problem—the temperature depends only on x and y, and satisfies the two-dimensional version of the Laplace equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

Suppose now, for instance, the temperature is specified on the boundary of our region D, and we wish to find the temperature $T(x,y)$ in region. We are simply looking for a solution of the Laplace equation that satisfies the specified boundary condition.

Let's look at another physical problem which leads to Laplace's equation. Gauss's Law of electrostatics tells us that the integral over a closed surface S of the electric field **E** is proportional to the charge included in the region B enclosed by S. Thus in the absence of any charge, we have

$$\iint_S \mathbf{E} \cdot dA = 0.$$

But in this case, we know the field **E** is conservative; let ϕ be the potential function—that is,

$$\mathbf{E} = \nabla \varphi.$$

Thus,

$$\iint_S \mathbf{E} \cdot dA = \iint_S \nabla \varphi \cdot dA.$$

Again, we call on the Divergence Theorem to conclude that φ must satisfy the Laplace equation. Mathematically, we cannot tell the problem of finding the electric potential in a region D, given the potential on the boundary of D, from the previous problem of finding the temperature in the region, given the temperature on the boundary. These are but two of the many physical problems that lead to the Laplace equation—You probably already know of some others. Let D be a domain and let σ be a given function continuous on the boundary of D. The problem of finding a function φ harmonic on the interior of D and which agrees with σ on the boundary of D is called the **Dirichlet problem.**

7.2. Harmonic functions. If D is a region in the plane, a real-valued function $u(x,y)$ having continuous second partial derivatives is said to be **harmonic** on D if it satisfies Laplace's equation on D :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

There is an intimate relationship between harmonic functions and analytic functions. Suppose f is analytic on D, and let $f(z) = u(x,y) + iv(x,y)$. Now, from the Cauchy-Riemann equations, we know

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \text{ and}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

If we differentiate the first of these with respect to x and the second with respect to y, and then add the two results, we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0.$$

Thus the real part of any analytic function is harmonic! Next, if we differentiate the first of the Cauchy-Riemann equations with respect to y and the second with respect to x, and then subtract the second from the first, we have

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0,$$

and we see that the imaginary part of an analytic function is also harmonic.

There is even more excitement. Suppose we are given a function φ harmonic in a *simply connected* region D. Then there is a function f analytic on D which is such that $\mathrm{Re} f = \varphi$. Let's see why this is so. First, define g by

$$g(z) = \frac{\partial\varphi}{\partial x} - i\frac{\partial\varphi}{\partial y}.$$

We'll show that g is analytic by verifying that the real and imaginary parts satisfy the Cauchy-Riemann equations:

$$\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right) = \frac{\partial^2\varphi}{\partial x^2} = -\frac{\partial^2\varphi}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{\partial\varphi}{\partial y}\right),$$

since φ is harmonic. Next,

$$\frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial x}\right) = \frac{\partial^2\varphi}{\partial y\partial x} = \frac{\partial^2\varphi}{\partial x\partial y} = -\frac{\partial}{\partial x}\left(-\frac{\partial\varphi}{\partial y}\right).$$

Since g is analytic on the simply connected region D, we know that the integral of g around any closed curve is zero, and so it has an antiderivative $G(z) = u + iv$. This antiderivative is, of course, analytic on D, and we know that

$$G'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial\varphi}{\partial x} - i\frac{\partial\varphi}{\partial y}.$$

Thus, $u(x,y) = \varphi(x,y) + h(y)$. From this,

$$\frac{\partial u}{\partial y} = \frac{\partial\varphi}{\partial y} + h'(y),$$

and so $h'(y) = 0$, or h = constant, from which it follows that $u(x,y) = \varphi(x,y) + c$. In other words, $\operatorname{Re} G = u$, as we promised to show.

Example

The function $\varphi(x,y) = x^3 - 3xy^2$ is harmonic everywhere. We shall find an analytic function G so that $\operatorname{Re} G = \varphi$. We know that $G(z) = (x^3 - 3xy^2) + iv$, and so from the Cauchy-Riemann equations:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6xy$$

Hence,

$$v(x,y) = 3x^2y + k(y).$$

To find $k(y)$ differentiate with respect to y :

$$\frac{\partial v}{\partial y} = 3x^2 + k'(y) = \frac{\partial u}{\partial x} = 3x^2 - 3y^2,$$

and so,

$$k'(y) = -3y^2, \text{ or}$$
$$k(y) = -y^3 + \text{ any constant.}$$

If we choose the constant to be zero, this gives us

$$v = 3x^2y + k(y) = 3x^2y - y^3,$$

and finally,

$$G(z) = u + iv = (x^3 - 3xy^2) + i(3x^2y - y^3).$$

Exercises

1. Suppose φ is harmonic on a simply connected region D. Prove that if φ assumes its maximum or its minimum value at some point in D, then φ is constant in D.

2. Suppose φ and σ are harmonic in a simply connected region D bounded by the curve C. Suppose moreover that $\varphi(x,y) = \sigma(x,y)$ for all $(x,y) \in C$. Explain how you know that $\varphi = \sigma$ everywhere in D.

3. Find an entire function f such that $\mathrm{Re} f = x^2 - 3x - y^2$, or explain why there is no such function f.

4. Find an entire function f such that $\mathrm{Re} f = x^2 + 3x - y^2$, or explain why there is no such function f.

7.3. Poisson's integral formula. Let Λ be the disk bounded by the circle $C_\rho = \{z : |z| = \rho\}$. Suppose φ is harmonic on Λ and let f be a function analytic on Λ and such that $\mathrm{Re} f = \varphi$. Now then, for fixed z with $|z| < \rho$, the function

$$g(s) = \frac{f(s)\overline{z}}{\rho^2 - s\overline{z}}$$

is analytic on Λ. Thus from Cauchy's Theorem

$$\int_{C_\rho} g(s)ds = \int_{C_\rho} \frac{f(s)\overline{z}}{\rho^2 - s\overline{z}} ds = 0.$$

We know also that

$$f(z) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(s)}{s-z} ds.$$

Adding these two equations gives us

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_\rho} \left(\frac{1}{s-z} + \frac{\overline{z}}{\rho^2 - s\overline{z}} \right) f(s)ds \\ &= \frac{1}{2\pi i} \int_{C_\rho} \frac{\rho^2 - |z|^2}{(s-z)(\rho^2 - s\overline{z})} f(s)ds. \end{aligned}$$

Next, let $\gamma(t) = \rho e^{it}$, and our integral becomes

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{\rho^2 - |z|^2}{(\rho e^{it} - z)(\rho^2 - \rho e^{it}\overline{z})} f(\rho e^{it}) i\rho e^{it} dt \\ &= \frac{\rho^2 - |z|^2}{2\pi} \int_0^{2\pi} \frac{f(\rho e^{it})}{(\rho e^{it} - z)(\rho e^{-it} - \overline{z})} dt \\ &= \frac{\rho^2 - |z|^2}{2\pi} \int_0^{2\pi} \frac{f(\rho e^{it})}{|\rho e^{it} - z|^2} dt \end{aligned}$$

Now,

$$\varphi(x,y) = \mathrm{Re} f = \frac{\rho^2 - |z|^2}{2\pi} \int_0^{2\pi} \frac{\varphi(\rho e^{it})}{|\rho e^{it} - z|^2} dt.$$

Next, use polar coordinates: $z = re^{i\theta}$:

$$\varphi(r,\theta) = \frac{\rho^2 - r^2}{2\pi} \int_0^{2\pi} \frac{\varphi(\rho e^{it})}{|\rho e^{it} - re^{i\theta}|^2} dt.$$

Now,

$$\begin{aligned} |\rho e^{it} - re^{i\theta}|^2 &= (\rho e^{it} - re^{i\theta})(\rho e^{-it} - re^{-i\theta}) = \rho^2 + r^2 - r\rho(e^{i(t-\theta)} + e^{-i(t-\theta)}) \\ &= \rho^2 + r^2 - 2r\rho\cos(t-\theta). \end{aligned}$$

Substituting this in the integral, we have **Poisson's integral formula**:

$$\varphi(r,\theta) = \frac{\rho^2 - r^2}{2\pi} \int_0^{2\pi} \frac{\varphi(\rho e^{it})}{\rho^2 + r^2 - 2r\rho\cos(t-\theta)} dt$$

This famous formula essentially solves the Dirichlet problem for a disk.

Exercises

5. Evaluate $\int_0^{2\pi} \frac{1}{\rho^2 + r^2 - 2r\rho\cos(t-\theta)} dt$. [Hint: This is easy.]

6. Suppose φ is harmonic in a region D. If $(x_0,y_0) \in D$ and if $C \subset D$ is a circle centered at (x_0,y_0), the inside of which is also in D, then $\varphi(x_0,y_0)$ is the average value of φ on the circle C.

7. Suppose φ is harmonic on the disk $\Lambda = \{z : |z| \leq \rho\}$. Prove that

$$\varphi(0,0) = \frac{1}{\pi\rho^2} \iint_\Lambda \varphi \, dA.$$