Chapter Eight

Series

8.1. Sequences. The basic definitions for complex sequences and series are essentially the same as for the real case. A sequence of complex numbers is a function $g : \mathbb{Z}_+ \to \mathbb{C}$ from the positive integers into the complex numbers. It is traditional to use subscripts to indicate the values of the function. Thus we write $g(n) = z_n$ and an explicit name for the sequence is seldom used; we write simply $(z_n)$ to stand for the sequence $g$ which is such that $g(n) = z_n$. For example, $(i^n)$ is the sequence $g$ for which $g(n) = i^n$.

The number $L$ is a limit of the sequence $(z_n)$ if given an $\varepsilon > 0$, there is an integer $N_\varepsilon$ such that $|z_n - L| < \varepsilon$ for all $n \geq N_\varepsilon$. If $L$ is a limit of $(z_n)$, we sometimes say that $(z_n)$ converges to $L$. We frequently write $\lim_{n \to \infty} z_n = L$. It is relatively easy to see that if the complex sequence $(z_n) = (u_n + iv_n)$ converges to $L$, then the two real sequences $(u_n)$ and $(v_n)$ each have a limit: $(u_n)$ converges to $\text{Re}L$ and $(v_n)$ converges to $\text{Im}L$. Conversely, if the two real sequences $(u_n)$ and $(v_n)$ each have a limit, then so also does the complex sequence $(u_n + iv_n)$. All the usual nice properties of limits of sequences are thus true:

$$
\lim (z_n \pm w_n) = \lim z_n \pm \lim w_n;
\lim (z_n w_n) = \lim z_n \lim w_n; \text{ and}
\lim \left( \frac{z_n}{w_n} \right) = \frac{\lim z_n}{\lim w_n}.
$$

provided that $\lim z_n$ and $\lim w_n$ exist. (And in the last equation, we must, of course, insist that $\lim w_n \neq 0$.)

A necessary and sufficient condition for the convergence of a sequence $(a_n)$ is the celebrated Cauchy criterion: given $\varepsilon > 0$, there is an integer $N_\varepsilon$ so that $|a_n - a_m| < \varepsilon$ whenever $n, m > N_\varepsilon$.

A sequence $(f_n)$ of functions on a domain $D$ is the obvious thing: a function from the positive integers into the set of complex functions on $D$. Thus, for each $z \in D$, we have an ordinary sequence $(f_n(z))$. If each of the sequences $(f_n(z))$ converges, then we say the sequence of functions $(f_n)$ converges to the function $f$ defined by $f(z) = \lim f_n(z))$. This pretty obvious stuff. The sequence $(f_n)$ is said to converge to $f$ uniformly on a set $S$ if given an $\varepsilon > 0$, there is an integer $N_\varepsilon$ so that $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq N_\varepsilon$ and all $z \in S$.

Note that it is possible for a sequence of continuous functions to have a limit function that is not continuous. This cannot happen if the convergence is uniform. To see this, suppose the sequence $(f_n)$ of continuous functions converges uniformly to $f$ on a domain $D$, let $z_0 \in D$, and let $\varepsilon > 0$. We need to show there is a $\delta$ so that $|f(z_0) - f(z)| < \varepsilon$ whenever
Let’s do it. First, choose \( N \) so that \( |f_N(z) - f(z)| < \frac{\varepsilon}{3} \). We can do this because of the uniform convergence of the sequence \((f_n)\). Next, choose \( \delta \) so that \( |f_N(z_0) - f_N(z)| < \frac{\varepsilon}{3} \) whenever \( |z_0 - z| < \delta \). This is possible because \( f_N \) is continuous. Now then, when \( |z_0 - z| < \delta \), we have

\[
|f(z_0) - f(z)| = |f(z_0) - f_N(z_0) + f_N(z_0) - f_N(z) + f_N(z) - f(z)| \\
\leq |f(z_0) - f_N(z_0)| + |f_N(z_0) - f_N(z)| + |f_N(z) - f(z)| \\
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\]

and we have done it!

Now suppose we have a sequence \((f_n)\) of continuous functions which converges uniformly on a contour \( C \) to the function \( f \). Then the sequence \( \int_C f_n(z) \, dz \) converges to \( \int_C f(z) \, dz \). This is easy to see. Let \( \varepsilon > 0 \). Now let \( N \) be so that \( |f_n(z) - f(z)| < \frac{\varepsilon}{A} \) for \( n > N \), where \( A \) is the length of \( C \). Then,

\[
\left| \int_C f_n(z) \, dz - \int_C f(z) \, dz \right| = \left| \int_C (f_n(z) - f(z)) \, dz \right| \\
< \frac{\varepsilon}{A} A = \varepsilon
\]

whenever \( n > N \).

Now suppose \((f_n)\) is a sequence of functions each \emph{analytic} on some region \( D \), and suppose the sequence converges uniformly on \( D \) to the function \( f \). Then \( f \) is analytic. This result is in marked contrast to what happens with real functions—examples of uniformly convergent sequences of differentiable functions with a nondifferentiable limit abound in the real case. To see that this uniform limit is analytic, let \( z_0 \in D \), and let \( S = \{z : |z - z_0| < r\} \subset D \). Now consider any simple closed curve \( C \subset S \). Each \( f_n \) is analytic, and so \( \int_C f_n(z) \, dz = 0 \) for every \( n \). From the uniform convergence of \((f_n)\), we know that \( \int_C f(z) \, dz \) is the limit of the sequence \( \left( \int_C f_n(z) \, dz \right) \), and so \( \int_C f(z) \, dz = 0 \). Morera’s theorem now tells us that \( f \) is analytic on \( S \), and hence at \( z_0 \). Truly a miracle.

\section*{Exercises}

8.2
1. Prove that a sequence cannot have more than one limit. (We thus speak of the limit of a sequence.)

2. Give an example of a sequence that does not have a limit, or explain carefully why there is no such sequence.

3. Give an example of a bounded sequence that does not have a limit, or explain carefully why there is no such sequence.

4. Give a sequence \((f_n)\) of functions continuous on a set \(D\) with a limit that is not continuous.

5. Give a sequence of real functions differentiable on an interval which converges uniformly to a nondifferentiable function.

8.2 Series. A series is simply a sequence \((s_n)\) in which \(s_n = a_1 + a_2 + \ldots + a_n\). In other words, there is sequence \((a_n)\) so that \(s_n = s_{n-1} + a_n\). The \(s_n\) are usually called the partial sums. Recall from Mrs. Turner’s class that if the series \(\sum_{j=1}^{\infty} a_j\) has a limit, then it must be true that \(\lim_{n \to \infty} (a_n) = 0\).

Consider a series \(\left(\sum_{j=1}^{n} f_j(z)\right)\) of functions. Chances are this series will converge for some values of \(z\) and not converge for others. A useful result is the celebrated Weierstrass M-test: Suppose \((M_j)\) is a sequence of real numbers such that \(M_j \geq 0\) for all \(j > J\), where \(J\) is some number, and suppose also that the series \(\sum_{j=1}^{n} M_j\) converges. If for all \(z \in D\), we have \(|f_j(z)| \leq M_j\) for all \(j > J\), then the series \(\sum_{j=1}^{n} f_j(z)\) converges uniformly on \(D\).

To prove this, begin by letting \(\varepsilon > 0\) and choosing \(N > J\) so that

\[\sum_{j=m}^{n} M_j < \varepsilon\]

for all \(n, m > N\). (We can do this because of the famous Cauchy criterion.) Next, observe that
\[
\left| \sum_{j=m}^{n} f_j(z) \right| \leq \sum_{j=m}^{n} |f_j(z)| \leq \sum_{j=m}^{n} M_j < \varepsilon.
\]

This shows that \( \left( \sum_{j=1}^{n} f_j(z) \right) \) converges. To see the uniform convergence, observe that
\[
\left| \sum_{j=m}^{n} f_j(z) \right| = \left| \sum_{j=0}^{n} f_j(z) - \sum_{j=0}^{m-1} f_j(z) \right| < \varepsilon
\]
for all \( z \in D \) and \( n > m > N \). Thus,
\[
\lim_{n \to \infty} \left| \sum_{j=0}^{n} f_j(z) - \sum_{j=0}^{m-1} f_j(z) \right| = \left| \sum_{j=0}^{\infty} f_j(z) - \sum_{j=0}^{m-1} f_j(z) \right| \leq \varepsilon
\]
for \( m > N \). (The limit of a series \( \left( \sum_{j=0}^{n} a_j \right) \) is almost always written as \( \sum_{j=0}^{\infty} a_j \).)

**Exercises**

6. Find the set \( D \) of all \( z \) for which the sequence \( \left( \frac{z^n}{z^n-3^n} \right) \) has a limit. Find the limit.

7. Prove that the series \( \left( \sum_{j=1}^{n} a_j \right) \) converges if and only if both the series \( \left( \sum_{j=1}^{n} \text{Re} a_j \right) \) and \( \left( \sum_{j=1}^{n} \text{Im} a_j \right) \) converge.

8. Explain how you know that the series \( \left( \sum_{j=1}^{n} \left( \frac{1}{z} \right)^j \right) \) converges uniformly on the set \( |z| \geq 5 \).

8.3 **Power series.** We are particularly interested in series of functions in which the partial sums are polynomials of increasing degree:
\[
s_n(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \ldots + c_n(z - z_0)^n.
\]
(We start with $n = 0$ for esthetic reasons.) These are the so-called **power series**. Thus, a power series is a series of functions of the form \( \sum_{j=0}^{n} c_j (z - z_0)^j \).

Let’s look first at a very special power series, the so-called **Geometric series**: \[
\sum_{j=0}^{n} z^j.
\]

Here
\[
s_n = 1 + z + z^2 + \ldots + z^n, \quad \text{and} \quad z s_n = z + z^2 + z^3 + \ldots + z^{n+1}.
\]

Subtracting the second of these from the first gives us
\[
(1 - z) s_n = 1 - z^{n+1}.
\]

If $z = 1$, then we can’t go any further with this, but I hope it’s clear that the series does not have a limit in case $z = 1$. Suppose now $z \neq 1$. Then we have
\[
s_n = \frac{1}{1 - z} - \frac{z^{n+1}}{1 - z}.
\]

Now if $|z| < 1$, it should be clear that $\lim (z^{n+1}) = 0$, and so
\[
\lim \left( \sum_{j=0}^{n} z^j \right) = \lim s_n = \frac{1}{1 - z}.
\]

Or,
\[
\sum_{j=0}^{\infty} z^j = \frac{1}{1 - z}, \quad \text{for} \ |z| < 1.
\]

There is a bit more to the story. First, note that if $|z| > 1$, then the Geometric series does not have a limit (why?). Next, note that if $|z| \leq \rho < 1$, then the Geometric series converges
uniformly to $\frac{1}{1-z}$. To see this, note that

$$\left(\sum_{j=0}^{n} \rho^j \right)$$

has a limit and appeal to the Weierstrass M-test.

Clearly a power series will have a limit for some values of $z$ and perhaps not for others. First, note that any power series has a limit when $z = z_0$. Let’s see what else we can say. Consider a power series $\left(\sum_{j=0}^{n} c_j(z-z_0)^j \right)$. Let

$$\lambda = \lim \sup \left( \sqrt{|c_j|} \right).$$

(Recall from 6th grade that $\lim \sup (a_k) = \lim (\sup \{a_k : k \geq n\})$.) Now let $R = \frac{1}{\lambda}$. (We shall say $R = 0$ if $\lambda = \infty$, and $R = \infty$ if $\lambda = 0$.) We are going to show that the series converges uniformly for all $|z-z_0| \leq \rho < R$ and diverges for all $|z-z_0| > R$.

First, let’s show the series does not converge for $|z-z_0| > R$. To begin, let $k$ be so that

$$\frac{1}{|z-z_0|} < k < \frac{1}{R} = \lambda.$$

There are an infinite number of $c_j$ for which $\sqrt{|c_j|} > k$, otherwise $\lim \sup \left( \sqrt{|c_j|} \right) \leq k$. For each of these $c_j$ we have

$$|c_j(z-z_0)^j| = \left( \sqrt{|c_j|} |z-z_0| \right)^j > (k|z-z_0|)^j > 1.$$ 

It is thus not possible for $\lim_{n \to \infty} |c_n(z-z_0)^n| = 0$, and so the series does not converge.

Next, we show that the series does converge uniformly for $|z-z_0| \leq \rho < R$. Let $k$ be so that

$$\lambda = \frac{1}{R} < k < \frac{1}{\rho}.$$ 

Now, for $j$ large enough, we have $\sqrt{|c_j|} < k$. Thus for $|z-z_0| \leq \rho$, we have
The geometric series \( \sum_{j=0}^{n} (k\rho)^j \) converges because \( k\rho < 1 \) and the uniform convergence of \( \sum_{j=0}^{n} c_j(z-z_0)^j \) follows from the M-test.

**Example**

Consider the series \( \sum_{j=0}^{n} \frac{1}{j!} z^j \). Let’s compute \( R = \frac{1}{\limsup \sqrt[n]{|c_j|}} = \limsup \sqrt[n]{|j!|} \). Let \( K \) be any positive integer and choose an integer \( m \) large enough to insure that \( 2^m > \frac{K^{2K}}{(2K)!} \). Now consider \( \frac{n!}{K^n} \), where \( n = 2K + m \):

\[
\frac{n!}{K^n} = \frac{(2K + m)!}{K^{2K+m}} = \frac{(2K + m)(2K + m - 1) \ldots (2K + 1)(2K)!}{K^mK^{2K}} > 2^m \frac{(2K)!}{K^{2K}} > 1
\]

Thus \( \sqrt[n]{n!} > K \). Reflect on what we have just shown: given any number \( K \), there is a number \( n \) such that \( \sqrt[n]{n!} \) is bigger than it. In other words, \( R = \limsup \sqrt[n]{|j!|} = \infty \), and so the series \( \sum_{j=0}^{n} \frac{1}{j!} z^j \) converges for all \( z \).

Let’s summarize what we have. For any power series \( \sum_{j=0}^{n} c_j(z-z_0)^j \), there is a number

\[
R = \limsup \sqrt[n]{|c_j|}
\]

such that the series converges uniformly for \( |z-z_0| \leq \rho < R \) and does not converge for \( |z-z_0| > R \). (Note that we may have \( R = 0 \) or \( R = \infty \).) The number \( R \) is called the **radius of convergence** of the series, and the set \( |z-z_0| = R \) is called the **circle of convergence**. Observe also that the limit of a power series is a function analytic inside the circle of convergence (why?).

**Exercises**

9. Suppose the sequence of real numbers \( (a_j) \) has a limit. Prove that
\[
\lim \sup(\alpha_j) = \lim(\alpha_j).
\]

For each of the following, find the set \( D \) of points at which the series converges:

10. \( \sum_{j=0}^{n} j! z^j \).

11. \( \sum_{j=0}^{n} jz^j \).

12. \( \sum_{j=0}^{n} \frac{j^2}{3^j} z^j \).

13. \( \sum_{j=0}^{n} \frac{(-1)^j}{2^j(j!)^2} z^j \).

8.4 Integration of power series. Inside the circle of convergence, the limit

\[
S(z) = \sum_{j=0}^{\infty} c_j(z - z_0)^j
\]

is an analytic function. We shall show that this series may be integrated "term-by-term"—that is, the integral of the limit is the limit of the integrals. Specifically, if \( C \) is any contour inside the circle of convergence, and the function \( g \) is continuous on \( C \), then

\[
\int_{C} g(z)S(z)dz = \sum_{j=0}^{\infty} c_j \int_{C} g(z)(z - z_0)^j dz.
\]

Let’s see why this. First, let \( \varepsilon > 0 \). Let \( M \) be the maximum of \(|g(z)|\) on \( C \) and let \( L \) be the length of \( C \). Then there is an integer \( N \) so that

\[
\left| \sum_{j=N}^{\infty} c_j(z - z_0)^j \right| < \frac{\varepsilon}{ML}
\]
for all $n > N$. Thus,

$$\left| \int_{C} g(z) \sum_{j=n}^{\infty} c_j(z-z_0)^j \, dz \right| < ML \cdot \frac{\varepsilon}{ML} = \varepsilon,$$

Hence,

$$\left| \int_{C} g(z)S(z)dz - \sum_{j=0}^{n-1} c_j \int_{C} g(z)(z-z_0)^j/\, dz \right| = \left| \int_{C} g(z) \sum_{j=n}^{\infty} c_j(z-z_0)^j \, dz \right| < \varepsilon,$$

and we have shown what we promised.

### 8.5 Differentiation of power series

Again, let

$$S(z) = \sum_{j=0}^{\infty} c_j(z-z_0)^j.$$  

Now we are ready to show that inside the circle of convergence,

$$S'(z) = \sum_{j=1}^{\infty} j c_j(z-z_0)^{j-1}.$$  

Let $z$ be a point inside the circle of convergence and let $C$ be a positive oriented circle centered at $z$ and inside the circle of convergence. Define

$$g(s) = \frac{1}{2\pi i(s-z)^2},$$

and apply the result of the previous section to conclude that

8.9
\[ \int_{C} g(s)S(s)\, ds = \sum_{j=0}^{\infty} c_j \int_{C} g(s)(s - z_0)^j\, ds, \text{ or} \]
\[ \frac{1}{2\pi i} \int_{C} \frac{S(s)}{(s - z)^2}\, ds = \sum_{j=0}^{\infty} c_j \frac{1}{2\pi i} \int_{C} \frac{(s - z_0)^j}{(s - z)^2}\, ds. \] Thus
\[ S'(z) = \sum_{j=0}^{\infty} j c_j (z - z_0)^{j-1}, \]
as promised!

**Exercises**

14. Find the limit of
\[ \left( \sum_{j=0}^{n} (j + 1)z^j \right). \]
For what values of \( z \) does the series converge?

15. Find the limit of
\[ \left( \sum_{j=1}^{n} \frac{z^j}{j} \right). \]
For what values of \( z \) does the series converge?

16. Find a power series \( \sum_{j=0}^{n} c_j(z - 1)^j \) such that
\[ \frac{1}{z} = \sum_{j=0}^{\infty} c_j(z - 1)^j, \text{ for } |z - 1| < 1. \]

17. Find a power series \( \sum_{j=0}^{n} c_j(z - 1)^j \) such that
\[
\log z = \sum_{j=0}^{\infty} c_j (z - 1)^j, \text{ for } |z - 1| < 1.
\]