Chapter Nine

Taylor and Laurent Series

9.1. **Taylor series.** Suppose $f$ is analytic on the open disk $|z - z_0| < r$. Let $z$ be any point in this disk and choose $C$ to be the positively oriented circle of radius $\rho$, where $|z - z_0| < \rho < r$. Then for $s \in C$ we have

\[
\frac{1}{s-z} = \frac{1}{(s-z_0) - (z-z_0)} = \frac{1}{s-z_0} \left[ \frac{1}{1 - \frac{s-z_0}{s-z_0}} \right] = \sum_{j=0}^{\infty} \left( \frac{z-z_0}{s-z_0} \right)^j
\]

since $\left| \frac{z-z_0}{s-z_0} \right| < 1$. The convergence is uniform, so we may integrate

\[
\int_{C} \frac{f(s)}{s-z} \, ds = \sum_{j=0}^{\infty} \left( \frac{1}{2\pi i} \int_{C} \frac{f(s)}{(s-z_0)^{j+1}} \, ds \right) \left( z-z_0 \right)^j, \text{ or}
\]

\[
f(z) = \frac{1}{2\pi i} \int_{C} \frac{f(s)}{s-z} \, ds = \sum_{j=0}^{\infty} \left( \frac{1}{2\pi i} \int_{C} \frac{f(s)}{(s-z_0)^{j+1}} \, ds \right) \left( z-z_0 \right)^j.
\]

We have thus produced a power series having the given analytic function as a limit:

\[
f(z) = \sum_{j=0}^{\infty} c_j (z-z_0)^j, \quad |z-z_0| < r,
\]

where

\[
c_j = \frac{1}{2\pi i} \int_{C} \frac{f(s)}{(s-z_0)^{j+1}} \, ds.
\]

This is the celebrated **Taylor Series** for $f$ at $z = z_0$.

We know we may differentiate the series to get

\[
f'(z) = \sum_{j=1}^{\infty} j c_j (z-z_0)^{j-1}
\]
and this one converges uniformly where the series for $f$ does. We can thus differentiate again and again to obtain

$$f^{(n)}(z) = \sum_{j=n}^{\infty} j(j-1)(j-2)\ldots(j-n+1)c_j(z-z_0)^{j-n}. $$

Hence,

$$f^{(n)}(z_0) = n!c_n, \text{ or } c_n = \frac{f^{(n)}(z_0)}{n!}.$$

But we also know that

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z_0)^{n+1}} ds. $$

This gives us

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(s)}{(s-z_0)^{n+1}} ds, \text{ for } n = 0, 1, 2, \ldots.$$

This is the famous **Generalized Cauchy Integral Formula**. Recall that we previously derived this formula for $n = 0$ and 1.

What does all this tell us about the radius of convergence of a power series? Suppose we have

$$f(z) = \sum_{j=0}^{\infty} c_j(z-z_0)^j,$$

and the radius of convergence is $R$. Then we know, of course, that the limit function $f$ is analytic for $|z-z_0| < R$. We showed that if $f$ is analytic in $|z-z_0| < r$, then the series converges for $|z-z_0| < r$. Thus $r \leq R$, and so $f$ cannot be analytic at any point $z$ for which $|z-z_0| > R$. In other words, the circle of convergence is the largest circle centered at $z_0$ inside of which the limit $f$ is analytic.
Example

Let \( f(z) = \exp(z) = e^z \). Then \( f(0) = f'(0) = \ldots = f^{(n)}(0) = \ldots = 1 \), and the Taylor series for \( f \) at \( z_0 = 0 \) is

\[
e^z = \sum_{j=0}^{\infty} \frac{1}{j!} z^j
\]

and this is valid for all values of \( z \) since \( f \) is entire. (We also showed earlier that this particular series has an infinite radius of convergence.)

Exercises

1. Show that for all \( z \),

\[
e^z = e \sum_{j=0}^{\infty} \frac{1}{j!} (z - 1)^j.
\]

2. What is the radius of convergence of the Taylor series \( \left( \sum_{j=0}^{n} c_j z^j \right) \) for \( \tanh z \)?

3. Show that

\[
\frac{1}{1 - z} = \sum_{j=0}^{\infty} \frac{(z - i)^j}{(1 - i)^{j+1}}
\]

for \( |z - i| < \sqrt{2} \).

4. If \( f(z) = \frac{1}{1 - z} \), what is \( f^{(10)}(i) \)?

5. Suppose \( f \) is analytic at \( z = 0 \) and \( f(0) = f'(0) = f''(0) = 0 \). Prove there is a function \( g \) analytic at 0 such that \( f(z) = z^3 g(z) \) in a neighborhood of 0.

6. Find the Taylor series for \( f(z) = \sin z \) at \( z_0 = 0 \).

7. Show that the function \( f \) defined by
\[
f(z) = \begin{cases} 
\frac{\sin z}{z} & \text{for } z \neq 0 \\
1 & \text{for } z = 0 
\end{cases}
\]

is analytic at \( z = 0 \), and find \( f'(0) \).

**9.2. Laurent series.** Suppose \( f \) is analytic in the region \( R_1 < |z - z_0| < R_2 \), and let \( C \) be a positively oriented simple closed curve around \( z_0 \) in this region. (Note: we include the possibilities that \( R_1 \) can be 0, and \( R_2 = \infty \).) We shall show that for \( z \notin C \) in this region

\[
f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j},
\]

where

\[
a_j = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{j+1}} ds, \text{ for } j = 0, 1, 2, \ldots
\]

and

\[
b_j = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{j+1}} ds, \text{ for } j = 1, 2, \ldots.
\]

The sum of the limits of these two series is frequently written

\[
f(z) = \sum_{j=-\infty}^{\infty} c_j (z - z_0)^j,
\]

where

\[
c_j = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{j+1}} ds, j = 0, \pm 1, \pm 2, \ldots.
\]

This recipe for \( f(z) \) is called a **Laurent series**, although it is important to keep in mind that it is really two series.

9.4
Okay, now let’s derive the above formula. First, let \( r_1 \) and \( r_2 \) be so that \( R_1 < r_1 \leq |z - z_0| \leq r_2 < R_2 \) and so that the point \( z \) and the curve \( C \) are included in the region \( r_1 \leq |z - z_0| \leq r_2 \). Also, let \( \Gamma \) be a circle centered at \( z \) and such that \( \Gamma \) is included in this region.

Then \( \frac{f(s)}{s - z} \) is an analytic function (of \( s \)) on the region bounded by \( C_1, C_2, \) and \( \Gamma \), where \( C_1 \) is the circle \( |z| = r_1 \) and \( C_2 \) is the circle \( |z| = r_2 \). Thus,

\[
\int_{C_2} \frac{f(s)}{s - z} \, ds = \int_{C_1} \frac{f(s)}{s - z} \, ds + \int_{\Gamma} \frac{f(s)}{s - z} \, ds.
\]

(All three circles are positively oriented, of course.) But \( \int_{\Gamma} \frac{f(s)}{s - z} \, ds = 2\pi if(z) \), and so we have

\[
2\pi if(z) = \int_{C_2} \frac{f(s)}{s - z} \, ds - \int_{C_1} \frac{f(s)}{s - z} \, ds.
\]

Look at the first of the two integrals on the right-hand side of this equation. For \( s \epsilon C_2 \), we have \( |z - z_0| < |s - z_0| \), and so

\[
\frac{1}{s - z} = \frac{1}{(s - z_0) - (z - z_0)} = \frac{1}{s - z_0} \left[ \frac{1}{1 - \left( \frac{z - z_0}{s - z_0} \right)} \right] = \frac{1}{s - z_0} \sum_{j=0}^{\infty} \left( \frac{z - z_0}{s - z_0} \right)^j = \sum_{j=0}^{\infty} \frac{1}{(s - z_0)^{j+1}} (z - z_0)^j.
\]
Hence,

\[
\int_{C_2} \frac{f(s)}{s - z} \, ds = \sum_{j=0}^{\infty} \left( \int_{C_2} \frac{f(s)}{(s - z_0)^{j+1}} \, ds \right) (z - z_0)^j
\]

\[
= \sum_{j=0}^{\infty} \left( \int_{C} \frac{f(s)}{(s - z_0)^{j+1}} \, ds \right) (z - z_0)^j
\]

For the second of these two integrals, note that for \( s \in C_1 \) we have \(|s - z_0| < |z - z_0|\), and so

\[
\frac{1}{s - z} = \frac{-1}{(z - z_0) - (s - z_0)} = \frac{-1}{z - z_0} \left[ \frac{1}{1 - \frac{s - z_0}{z - z_0}} \right]
\]

\[
= \frac{-1}{z - z_0} \sum_{j=0}^{\infty} \left( \frac{s - z_0}{z - z_0} \right)^j = -\sum_{j=0}^{\infty} (s - z_0)^j \frac{1}{(z - z_0)^{j+1}}
\]

\[
= -\sum_{j=1}^{\infty} (s - z_0)^{j-1} \frac{1}{(z - z_0)^j} = -\sum_{j=1}^{\infty} \left( \frac{1}{(s - z_0)^{j+1}} \right) \frac{1}{(z - z_0)^j}
\]

As before,

\[
\int_{C_1} \frac{f(s)}{s - z} \, ds = -\sum_{j=1}^{\infty} \left( \int_{C_1} \frac{f(s)}{(s - z_0)^{j+1}} \, ds \right) \frac{1}{(z - z_0)^j}
\]

\[
= -\sum_{j=1}^{\infty} \left( \int_{C} \frac{f(s)}{(s - z_0)^{j+1}} \, ds \right) \frac{1}{(z - z_0)^j}
\]

Putting this altogether, we have the Laurent series:

\[
f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s - z} \, ds - \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{s - z} \, ds
\]

\[
= \sum_{j=0}^{\infty} \left( \frac{1}{2\pi i} \int_{C} \frac{f(s)}{(s - z_0)^{j+1}} \, ds \right) (z - z_0)^j + \sum_{j=1}^{\infty} \left( \frac{1}{2\pi i} \int_{C} \frac{f(s)}{(s - z_0)^{j+1}} \, ds \right) \frac{1}{(z - z_0)^j}.
\]

**Example**

9.6
Let $f$ be defined by

$$f(z) = \frac{1}{z(z-1)}.$$

First, observe that $f$ is analytic in the region $0 < |z| < 1$. Let’s find the Laurent series for $f$ valid in this region. First,

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}.$$

From our vast knowledge of the Geometric series, we have

$$f(z) = -\frac{1}{z} - \sum_{j=0}^{\infty} z^j.$$

Now let’s find another Laurent series for $f$, the one valid for the region $1 < |z| < \infty$. First,

$$\frac{1}{z-1} = \frac{1}{z} \left[ \frac{1}{1 - \frac{1}{z}} \right].$$

Now since $\left| \frac{1}{z} \right| < 1$, we have

$$\frac{1}{z-1} = \frac{1}{z} \sum_{j=0}^{\infty} z^{-j} = \sum_{j=1}^{\infty} z^{-j},$$

and so

$$f(z) = -\frac{1}{z} + \sum_{j=1}^{\infty} z^{-j} = -\frac{1}{z} + \sum_{j=1}^{\infty} z^{-j}$$

and

$$f(z) = \sum_{j=2}^{\infty} z^{-j}.$$

**Exercises**

8. Find two Laurent series in powers of $z$ for the function $f$ defined by
\begin{equation}
f(z) = \frac{1}{z^2(1-z)}
\end{equation}

and specify the regions in which the series converge to \( f(z) \).

9. Find two Laurent series in powers of \( z \) for the function \( f \) defined by

\begin{equation}
f(z) = \frac{1}{z(1+z^2)}
\end{equation}

and specify the regions in which the series converge to \( f(z) \).

10. Find the Laurent series in powers of \( z - 1 \) for \( f(z) = \frac{1}{z} \) in the region \( 1 < |z - 1| < \infty \).