1. Find all values of $i^i$.

\[ i^i = e^{i \log i} = e^{i(\log |i| + \arg i)} = e^{i(\pi/2 + 2k\pi)i} = e^{-(1+4k)\pi/2}, \text{ for } k = 0, \pm 1, \pm 2, \ldots. \]

2. Evaluate $\int_C (z + 2\pi)dz$, where $C$ is the path from $z = 0$ to $z = 1 + 2i$ consisting of the line segment from 0 to 1 together with the segment from 1 to 1 + 2i.

\[ \int_C (z + 2\pi)dz = \int_{L_1} (z + 2\pi)dz + \int_{L_2} (z + 2\pi)dz, \] where $L_1$ is the segment from 0 to 1, and $L_2$ is the segment from 1 to 1 + 2i.

A complex description of $L_1$ is simply $\gamma_1(t) = t$, for $0 \leq t \leq 1$. Thus, 

\[ \int_{L_1} (z + 2\pi)dz = \int_0^1 (\gamma_1(t) + 2\gamma_1(t))\gamma_1'(t)dt = \int_0^1 (t + 2t)dt = \frac{3}{2}t^2 \bigg|_0^1 = \frac{3}{2}. \]

A complex description of $L_2$ is $\gamma_2(t) = 1 + 2ti$, for $0 \leq t \leq 1$. Thus, 

\[ \int_{L_2} (z + 2\pi)dz = \int_0^1 (\gamma_2(t) + 2\gamma_2(t))\gamma_2'(t)dt = \int_0^1 (1 + 2ti + 2 - 4ti)2idt \]

\[ = \int_0^1 (6i + 4t)dt = 6i + 2. \]

Hence, 

\[ \int_C (z + 2\pi)dz = \int_{L_1} (z + 2\pi)dz + \int_{L_2} (z + 2\pi)dz = \frac{7}{2} + 6i. \]

3. Show that the function $f$ defined by 

\[ f(z) = \begin{cases} \frac{\sin z}{z} & \text{for } z \neq 0 \\ 1 & \text{for } z = 0 \end{cases} \]

is analytic at $z = 0$, and find the derivative $f'(0)$.

For all $z$, we have $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$. Thus, 

\[ \frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots, \]

and we see that 

\[ f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots \]
is the limit of a power series valid for all $z$. Hence $f$ is, in fact, entire, and we see that $f'(0) = 0$.

4. Let

$$f(z) = \frac{1}{z^2(z + 2i)}.$$

a) Find a Laurent series in powers of $z$ which converges to $f$ and specify the region on which the series converges.

$$\frac{1}{z^2(z + 2i)} = \frac{1}{z^2} \cdot \frac{1}{(z + 2i)}$$

Now,

$$\frac{1}{z + 2i} = \frac{1}{2i} \left[ \frac{1}{1 + \frac{z}{2i}} \right] = \frac{1}{2i} \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{(2i)^k}, \text{ for } \left| \frac{z}{2i} \right| < 1, \text{ or } |z| < 2.$$  

We now have the Laurent series

$$f(z) = \frac{1}{z^2} \frac{1}{2i} \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{(2i)^k} = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{(2i)^k} = \sum_{k=-2}^{\infty} \frac{(-1)^k z^k}{(2i)^{k+3}}, \text{ valid for } 0 < |z| < 2.$$  

b) Find another Laurent series in powers of $z$ which converges to $f$ and specify the region on which the series converges.

Here consider,

$$\frac{1}{z + 2i} = \frac{1}{z} \left[ \frac{1}{1 + \frac{2i}{z}} \right] = \frac{1}{z} \sum_{k=0}^{\infty} \frac{(-1)^k (2i)^k}{z^k} = \sum_{k=0}^{\infty} \frac{(-1)^k (2i)^k}{z^k} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (2i)^{k-1}}{z^k}.$$  

This is valid for $|\frac{2i}{z}| < 1$, or $2 < |z|$. Then,

$$f(z) = \frac{1}{z^2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (2i)^{k-1}}{z^k} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (2i)^{k-1}}{z^{k+2}} = \sum_{k=3}^{\infty} \frac{(-2i)^{k-3}}{z^k}, \text{ for } 2 < |z|.$$  

5. Suppose $C$ is the circle $|z| = 5$ with the usual positive orientation. Evaluate the integrals:

a) $$\int_C \sin\left( \frac{1}{z} \right) dz.$$  

There is precisely one singular point inside $C$ and so the value of the integral is $2\pi i$ times the
residue at this point, $z = 0$. The Laurent expansion here is easy—it is simply

$$\sin \left( \frac{1}{z} \right) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \ldots$$

The residue is thus 1, and so

$$\int_C \sin \left( \frac{1}{z} \right) dz = 2\pi i$$

b) 

$$\int_C z \sin \left( \frac{1}{z} \right) dz.$$ 

The Laurent series at 0 is

$$z \sin \left( \frac{1}{z} \right) = 1 - \frac{1}{3!z^2} + \frac{1}{5!z^4} - \ldots,$$

and so the residue at 0 is 0. Hence,

$$\int_C z \sin \left( \frac{1}{z} \right) dz = 0.$$

c) 

$$\int_C z^2 \sin \left( \frac{1}{z} \right) dz.$$ 

Here we have

$$z^2 \sin \left( \frac{1}{z} \right) = z - \frac{1}{3!z^2} + \frac{1}{5!z^4} - \ldots,$$

and we see the residue is $\frac{-1}{3!} = -\frac{1}{6}$. Hence,

$$\int_C z^2 \sin \left( \frac{1}{z} \right) dz = 2\pi i \left( \frac{-1}{6} \right) = -\frac{\pi i}{3}.$$