Project 2 for Math 2605

This project is aimed at investigating the basins of attraction for Newton's method with two equations in two variables.

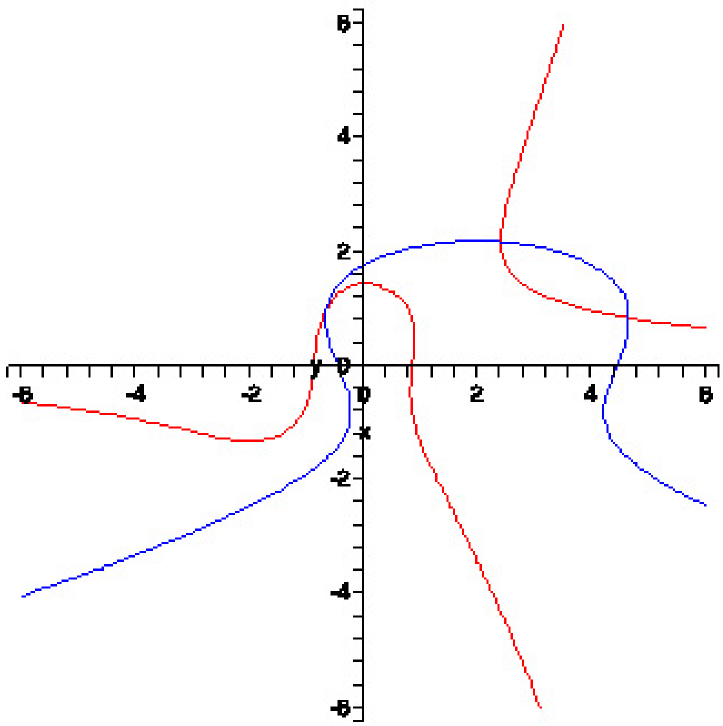
Consider the functions

$$f(x, y) = x^3y - 4x^2 - y^3 + 3$$

and

$$g(x, y) = x^2 + y^3 - 4x - 2y - 2\ .$$

Here is a graph showing the two curves defined implicitly by $f(x, y) = 0$ and $g(x, y) = 0$.



You see three solutions in the region shown, which is $-6 \leq x, y \leq 6$. Call the left most solution the *blue solution*, the right most solution the *green solution* and the middle solution the *red solution*.

Divide the square region $-6 \leq x, y \leq 6$ up into a 100 by 100 grid. Index the squares in a grid by i and j with $0 \leq i, j \leq 99$. The upper left hand corner of the i, jth square in the grid will be

$$(-6 + 12i/100, 6 - 12j/100)\ .$$

Run Newton's method for solving the system $f(x, y) = 0$ and $g(x, y) = 0$ staring from each of these 10,000 points. Label the starting point *blue*, *green*, *red* or *white* depending on whether you get convergence to the *blue*, *green* or *red* solutions, or none of the above. For this you will need to write a program that will implement Newton's method for this system. You can use whatever programing language you prefer for doing this, and this must be your own work.

To present and understand your results, produce a picture by coloring in the pixels of a 200 by 200 pixel square in which each square in your grid is shown as a 2 by 2 pixel square colored in with the corresponding color. Again, any way you want to do this is fine. You are free to seek assistance with the graphics here. If you use somebody else's help to do this, just credit them. You will not lose points for that.

Next, study your picture and answer the following questions. Present your answers after the graph.

Questions For each solution $\mathbf{z}$, determine from your graph the approximate radius r of the largest disk about $\mathbf{z}$ such that every starting point in the disk converges to $\mathbf{z}$. Measure it as closely as you can from your graph. What do you find? Can you explain the differences or similarities between these numbers? What do you notice about the boundary between the different basins of attraction?

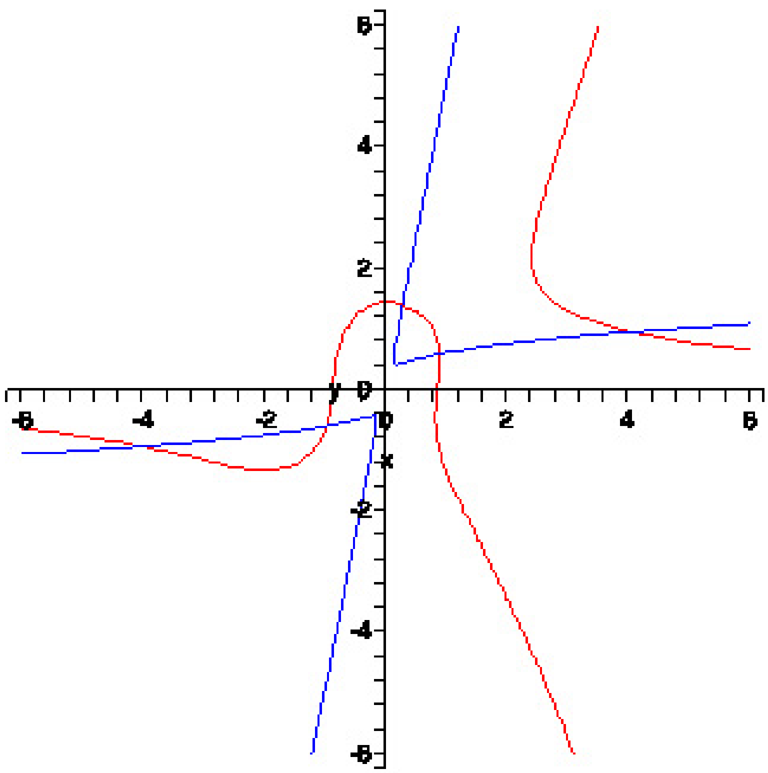
Now repeat this for the system $f(x,y) = 0$ and $g(x,y) = 0$ where now

$$f(x,y) = x^3y - 4x^2 - y^3 + 3$$

and

$$g(x,y) = x^2 + y^4 - 5xy^3 \ .$$

Only $g(x,y)$ has changed, and you will be able to use you coding with slight modifications. This time there are 5 solutions in the same region, so you will need more colors.



Apart from the changes consequent to this, proceed the same way, and answer the same questions.

Note on derivatives To run Newton's method, you have to take partial derivatives. There are no open source or even public–domain symbolic manipulation packages with the functionality of Maple, as far as I know. But even so they would probably be a bit slow.

The thing to do is to use the definition:

$$\frac{\partial}{\partial x} f(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \ .$$

This suggests that if we choose h rather small, say $h = 10^{-10}$, then

$$\frac{\partial}{\partial x} f(x,y) \approx \frac{f(x+10^{-10},y) - f(x,y)}{10^{-10}} = 10^{10}(f(x+10^{-10},y) - f(x,y)) \ .$$

The y derivatives can be handled in the same way, as can higher derivatives.

Extra Credit Implement this as a java applet in which f and g can be entered by the user, along with the region to be investigated. For extra credit that does not require writing an applet, consider the system $f(x,y) = 0$ and $g(x,y) = 0$ where

$$f(x,y) = x^3 y - 4x^2 - y^2 + 3$$

and

$$g(x,y) = x^2 + y^3 - 4x - 2y - 2 \ .$$

All that has changed from the first system considered is that the y^3 term in f has changed to y^2. If you plot the two curves implicitly defined by $f(x,y) = 0$ and $g(x,y) = 0$ in the region $-2 \le x \le 2$ and $0 \le y \le 2$, you see that they are nearly tangent to one another along a stretch between two solutions, near $x_0 = -0.5697$ and $y_0 = 1.200$. Produce a graph for this system and this region showing basisn of attraction, as above. How big are the basins of atteraction? (Use the radius of the largest circle inside the basin). Compute $J_{\mathbf{F}}$ for $\mathbf{F} = \begin{bmatrix} f \\ g \end{bmatrix}$ at (x_0, y_0). What does the size of this tell you? Can you see a relation between the size of $J_{\mathbf{F}}$ and the size of the basins of attractions?