CENTRAL LIMIT THEOREM FOR MAXWELLIAN MOLECULES AND TRUNCATION OF THE WILD EXPANSION

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Abstract: We prove an L^1 bound on the error made when the Wild summation for solutions of the Boltzmann equation for a gas of Maxwellian molecules is truncated at the nth stage. This gives quantitative control over the only constructive method known for solving the Boltzmann equation. As such, it has been recently applied to numerical computation, but without control on the approximation made in truncation. We also show that our bound is qualitatively sharp, and that it leads to a simple proof of the exponentially fast rate of relaxation to equilibrium for Maxwellian molecules, along lines originally suggested by McKean.

1. Introduction

The Boltzmann equation for the phase space density of a dilute gas

$$\frac{\partial}{\partial t} f(x,v,t) = -v \cdot \nabla_x f(x,v,t) + \frac{1}{\epsilon}\mathcal{Q}(f(x,v,t)) \tag{1.1}$$

contains terms accounting for the two ways that the density can change: The

$$-v \cdot \nabla_x f(x,v,t)$$

term represents the effects of *streaming*; that is, the motion

$$x_0 \mapsto x_0 + (t - t_0)v_0 \qquad v_0 \mapsto v_0 \tag{1.2}$$

of molecules between collisions. The $\mathcal{Q}(f(x,v,t))$ term represents the effects of binary collisions. The factor of $1/\epsilon$ in front of the collision kernel indicates that x and t are macroscopic coordinates, and ϵ is the ratio of a *kinetic length scale*, such as the mean free path, to a *macroscopic length scale*, such as a typical wavelength in the spatial density.

This paper concerns estimates on a constructive method for solving (1.1) in the spatially homogeneous case. The investigation of this constructive method is motivated in part by the so–called "splitting method" approach to the numerical solution of (1.1), and in part by the desire to better understand the spatially homogeneous case itself. While the rest of the paper will be focused on this latter case, a brief discussion of the splitting method approach to (1.1) will highlight the interest in studying constructive methods of solving the spatially homogeneous equation.

The splitting method works as follows: Fix a small time step Δt, and define two evolution operators as follows: The streaming operator $S_{\Delta t}$ is given by

$$S_{\Delta t} f(x,v) = f(x + v\Delta t, v)\ . \tag{1.3}$$

The collision operator $C_{\Delta t}$ is then defined by

$$C_{\Delta t} f(x,v) = f(x,v,\Delta t) \tag{1.4}$$

where $f(x,v,t)$ is the solution to

$$\frac{\partial}{\partial t} f(x,v,t) = \frac{1}{\epsilon}\mathcal{Q}(f(x,v,t)) \tag{1.5}$$

with initial data $f(x,v)$.

One then hopes that the solution to (1.1) is given by

$$f(x,v,t) = \lim_{n\to\infty} \big(C_{t/n} S_{t/n}\big)^n f_0(x,v) \tag{1.6}$$

and moreover, that the limit sets in fast so that we get a good approximation by taking n on the right hand side to be some reasonable number. Rigourous results of this type have recently been obtained by Desvillettes and Mischler [DM].

Given all of this, does one yet have a constructive method for generating approximate solutions to (1.1)? No, because while we have a simple and explicit formula for $S_{\Delta t}$, we do not have a general constructive method for computing $C_{\Delta t}$.

The equation (1.5), in which x is just a parameter, is really just the spatially homogeneous Boltzmann equation, and as such, it is indeed a much simpler equation than

(1.1). There are by now several proofs of the existence of solutions, and a great deal is known about the behavior of the solutions [D73][A88][We93], but there are in general no constructive methods for producing these solutions. This leaves the collision operator $C_{\Delta t}$ too implicit for practical applications involving actual computation.

However, when the colision operator $\mathcal{Q}$ is the one coressponding to the special case of Maxwellian molecules with cut-off, however, *there is a constructive method* for solving (1.5); namely, the Wild expansion [W51]. This has been applied by Gabetta, Pareschi and Toscani [GPT97] in the splitting approach to the numerical solution of (1.1) as described above.

The Wild expansion gives a convergent series expansion for the solution of (1.5). Each term in the expansion can be computed, and so if the series is truncated at some finite point, one gets a constructive approximation to $C_{\Delta t}$. This is the approximation that was used in [GPT97]. Our aim here is to quantitatively control the accuracy of this approximation.

Before stating our results, we need to explain a few things, starting with the explicit form of the collision kernel $\mathcal{Q}(f)$ in the case of Maxwellian molecules, to which our considerations are restricted.

For background information on the Boltzmann equation for Maxwellian molecules with cut-off (Maxwellian pseudomolecules), see Cercignani [C88]. What we take here as our starting point is that for Maxwellian molecules the collision kernel $\mathcal{Q}$ has the following form:

$$\mathcal{Q}\big(f\big)(v) = \kappa[f \circ f(v) - \rho f(v)]$$

where κ is a constant,

$$\rho = \int_{\mathbb{R}^3} f(v) \mathrm{d}v \ ,$$

and $f \circ g(v)$ denotes the *Wild convolution*

$$f \circ g(v) = \int_{\mathbf{R}^3 \times S^2} B\left(\cos\theta\right)\left[f(v^\star) g(w^\star)\right] \mathrm{d}w \mathrm{d}\mathbf{n} \ , \tag{1.7}$$

in which $\mathbf{n}$ is a unit vector, and $d\mathbf{n}$ denotes *normalized* surface measure on the unit sphere S^2. Also, $q = v - w$ is the relative velocity, and in $q \cdot \mathbf{n}$, the dot denotes the standard inner product. The vector $\mathbf{n}$ parameterizes the set of all kinematicly possible (those that conserve both energy and momentum) post–collisional velocities $v^\star$ and $w^\star$ by

$$v^\star = \frac{1}{2}(v + w + |q|\mathbf{n})$$
$$w^\star = \frac{1}{2}(v + w - |q|\mathbf{n})$$

and finally the scattering angle θ is given by $\cos\theta = q \cdot \mathbf{n}/|q|$. We have supressed the spatial coordinate x since it simply functions as a parameter in (1.5). We shall also rescale time to absorb the constant κ. This done, we have

$$\int_{S^2} B\left(\frac{q \cdot \mathbf{n}}{|q|}\right) \mathrm{d}\mathbf{n} = \frac{1}{2}\int_0^\pi B(\cos(\theta))\sin(\theta)\mathrm{d}\theta = 1 \tag{1.8}$$

so that the total mass ρ is conserved. Becuase of this, we may assume without loss of generality that our initial data for (1.5) is a probability density, and we shall do so throughout the rest of the paper. Also, we shall henceforth denote this initial data by F.

Wild [W51] observed that in this case, the initial value problem associated with (1.5),

$$\frac{\partial}{\partial t} f = f \circ f - f \qquad \text{and} \qquad f(\cdot, 0) = F(\cdot)\ ,$$

can be written as fixed point problem: Define the map $f \mapsto \Phi(f)$ by

$$\Phi(f)(t) = e^{-t}F + \int_0^t e^{-(t-s)} f \circ f(s)\mathrm{d}s \tag{1.9}$$

and observe that $f(t)$ solves (1.5) exactly when $\Phi(f) = f$.

To find fixed points, one iterates: Put $f_{(0)} = 0$, and then define

$$f_{(j+1)} = \Phi(f_{(j)}) \qquad \text{for all} \qquad j \geq 1 \tag{1.10}$$

This yields:

$$\begin{aligned}
f_{(1)}(t) =& e^{-t}F \\
f_{(2)}(t) =& e^{-t}F + e^{-t}(1-e^{-t})F \circ F \\
f_{(3)}(t) =& e^{-t}F + e^{-t}(1-e^{-t})F \circ F + \\
& e^{-t}(1-e^{-t})^2\big((1/2)F \circ (F \circ F) + (1/2)(F \circ F) \circ F\big)
\end{aligned}$$

and so on.

As one sees, $f_{(n)}(t) - f_{(n-1)}(t)$ is positive, so that $f_{(n)}(t) \geq f_{(n-1)}(t)$ and hence

$$\lim_{n\to\infty} f_{(n)}(t) = f(t) \tag{1.11}$$

exists, and as Wild showed [W51], is a solution of (1.8).

Wild's proof does not however provide any control on the difference between $f(t)$ and $f_{(n)}(t)$. The problem to be solved in this paper is the following:

Given $\epsilon > 0$, how large must one choose N to ensure that

$$\|f(t) - f_{(N)}(t)\|_{L^1} \leq \epsilon$$

uniformly in t, or on some interval $[0,T]$*?*

Actually, we shall work with

$$g_{(N)}(t) = f_{(N)}(t) + e^{-t}(1-e^{-t})^N M_F \tag{1.12}$$

where M_F denotes the Maxwellian density with the same mean and variance as F. We recall that the Maxwellian densities are the subset of Gaussian densities having isotropic covariance. We shall henceforth assume, without loss of generality, that F has zero mean and unit variance, so that

$$M_F(v) = \left(\frac{1}{2\pi}\right)^{3/2} e^{-|v|^2/2}$$

and we shall just write M in place of M_F.

The advantage of working with $g_{(N)}$ instead of $f_{(N)}$ is that $g_{(N)}(t)$ is a probability density for all N and t, as we shall see. That is, we add in the missing mass in Maxwellian form to make up for the fact that

$$\int_{\mathbb{R}^3} f_{(N)}(v,t)\mathrm{d}v = 1 - e^{-t}(1-e^{-t})^N \ .$$

We shall explain below why this Maxwellian choice is natural for making up the defect in the mass of $f_{(N)}$. In the meantime, we state our main result. This involves a smoothness assumption on the initial data involving Sobolev norms. Here, for any function f on $\mathbb{R}^3$ we define the H^k norm of f; i.e., the kth Sobolev norm $\|f\|_{\mathrm{H}^k}$, by

$$\|f\|^2_{\mathrm{H}^k} = \int |\xi|^{2k}|\widehat{f}(\xi)|^2\mathrm{d}\xi \ ,$$

where $\widehat{f}(\xi)$ denotes the Fourier transform of f:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^3} f(v)e^{-iv\cdot\xi}\mathrm{d}v \ .$$

The main result then is:

Theorem 1.1 *For any* $\delta > 0$ *and* $B < \infty$*, and all initial data* F *for (1.5) where* F *is a probability density with zero mean and unit variance satisfying*

$$\|F\|_{\mathrm{H}^{2+\delta}} + \int_{\mathbb{R}^3} |v|^{2+\delta}F(v)\mathrm{d}v \le B \ ,$$

one has

$$\|f(t) - g_{(N)}(t)\|_{L^1} \le (1-e^{-t})^N A b^{\ln N}$$

for all t, where $A < \infty$ and $b < 1$ depend only on B and δ.

Of course,

$$b^{\ln N} = N^{-|\ln b|}$$

so the bound given in Theorem 1.1 is algebraic in N. However, we shall see that what counts in the proof is $\ln N$ which measures the "generation" of the most recent collision in a typical collision history. For this reason, we have expressed our result in these terms. In section 5 we shall show that as a simple corollary to Theorem 1.1, one has that $\|f(t)-M(t)\|_{L^1}$ tends to zero exponentially fast in t. That there should be a proof of exponential convergence along these lines was suggested by McKean in [M66]. In the meantime, this exponential convergence in the L^1 norm for Maxwellian molecules was obtained for quite general initial data by Wennberg [We94]. However, the emphasis here is on bounding the error made in truncation of the Wild sum, so that one obtains a finitely computable approximation whose accuracy can be estimated, and estimated fairly closely. The fact that Theorem 1.1 leads to exponential convergence to the Maxwellian equilibrium, which is best possible, means that the result obtained here, algebraic in N, is best possible. However, while the constants in Theorem 1.1 are explicitly computable, we do not claim that they are best possible.

The proof of Theorem 1.1 has a number of ingredients, both probabilistic and analytic. In the next two subsections of this introduction, we introduce these and explain how they are combined to prove the main result.

1.1 Probabilistic lemmas

One of the main probabilistic ingredients is an explicit representation for $f_{(n)} - f_{(n-1)}$ in terms of F and a random walk on certain tree graphs. Analysis of this random walk will be central to our proof.

This random walk representation for $f_{(n)} - f_{(n-1)}$ was derived by McKean [M67]. To explain this, note that $f_{(n)} - f_{(n-1)}$ will be a weighted sum of n–fold Wild convolutions of F. While the Wild convolution is commutative, it is not associative [M66], and so there are many different n-fold Wild convolutions in general. To see this, consider two such objects for $n = 8$:

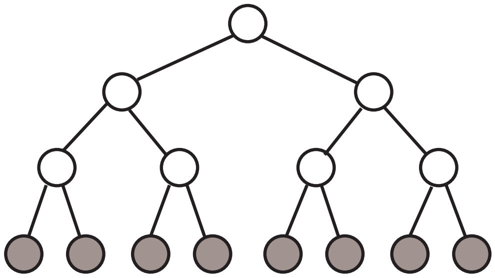
$$(((F \circ F) \circ (F \circ F)) \circ ((F \circ F) \circ (F \circ F))) \tag{1.13}$$

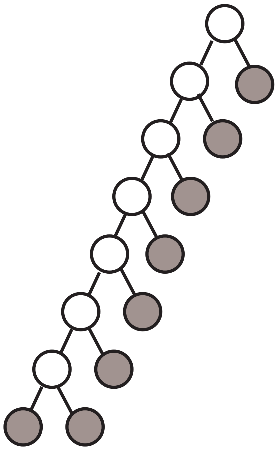
and

$$(((((((F \circ F) \circ F) \circ F) \circ F) \circ F) \circ F) \circ F) \tag{1.14}$$

The arrangement of the parentheses above is crucial because of the non–associativity of the Wild convolution.

There is a one–to–one correspondence, that we shall soon explain, between the different n-fold Wild convolutions and certain graphs. Let $\Gamma(n)$ denote the set of all binary tree graphs with n leaves and $n-1$ nodes in which *each node has either zero or two "children"; single "children" are not allowed.* We call $\cup_{n=2}^{\infty}\Gamma(n)$ the set of *McKean graphs.* We will always draw the graph so that the two "children" of any node can be identified as the "left child" and the "right child". This provides a natural left to right order on the nodes and leaves in any given "generation"; i.e., row. The particular McKean graphs corresponding to the two convolutions in (1.13) and (1.14) are





where the graph on the left corresponds to (1.13). Each of these graphs has 8 leaves, which are the shaded circles, and 7 nodes, which are the unshaded circles. The lines indicate the "parent–child" relationship of nodes and leaves, or nodes and other nodes: Each leaf, and each node other than the "root" node at the top, is the "child" of a "parent" node above it. The number of generations separating a leaf from the root is called the *depth* of the leaf. In the graphs as we are drawing them, the depth of a leaf is given by counting how many rows deep it lies, starting from 1 in the first row under the root. In the graph corresponding to (1.13), all of the leaves have the same depth, namely 3. In the graph corresponding to (1.14) there are two leaves of depth 7, and one each of every depth from 1 to 6. The depths of the leaves will be crucial in our analysis.

The correspondence between graphs γ and convolutions of F is the following: Pick any $\gamma \in \Gamma(n)$, and write "F" in each of the leaves. Now find the left–most pair of leaves in the deepest row. Erase this pair of leaves, which makes the former parent node a leaf, and write down $(F \circ F)$ in the new leaf. This produces a graph in $\Gamma(n-1)$. We then repeat the procedure, working our way back to the root, but with one difference: At the later steps, we will be erasing pairs of leaves that have various iterated convolutions of F written in them. *If the left leaf we are erasing has G_ℓ written in it, and the right leaf has G_r written in it, put $(G_\ell \circ G_r)$ in the new leaf created upon erasure.* Once one has done this until only the root is left, it has written in it some n–fold iterated convolution of F, and that is the convolution $C_\gamma(F)$ corresponding to the graph $\gamma \in \Gamma(n)$ that we started with. It should now be easy to check the two graphs in the diagram above do in fact lead to (1.13) and

(1.14) as claimed. We henceforth write $C_\gamma(F)$ denote the convolution corresponding to γ.

McKean's expression for $f_{(n)} - f_{(n-1)}$ is

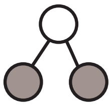
$$f_{(n)} - f_{(n-1)} = e^{-t}(1 - e^{-t})^{n-1}\left[\sum_{\gamma\in\Gamma(n)} p_n(\gamma)C_\gamma(F)\right] \tag{1.15}$$

where the $p_n(\gamma)$ are probabilities

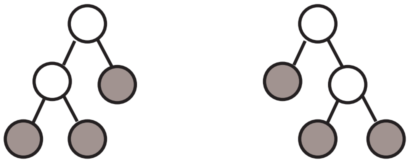
$$\sum_{\gamma\in\Gamma(n)} p_n(\gamma) = 1 \tag{1.16}$$

expressing the relative frequency with which any particular convolution $C_\gamma(F)$ contributes to $f_{(n)}-f_{(n-1)}$. Furthermore, McKean proved that these weights $p_n(\gamma)$ are the probabilities that a certain random walk on McKean graphs passes through the particular graph γ.

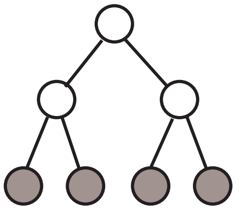
This random walk is the following process that builds a random McKean graph with n leaves in $n-2$ steps. Start with the unique element γ_0 in $\Gamma(2)$.



Pick a leaf at random – uniformly – and attach another copy of γ_0. Now there are three leaves; i.e., we get one of the two graphs in $\Gamma(3)$,



and each is equally likely. Pick one of these leaves at random, and again attach a copy of γ_0. There are 5 elements in $\Gamma(4)$, and this time they are not all equally likely: The graph



is produced with probability 1/3, while each of the others is produced with probability 1/6. This process is repeated until n leaves are produced. We call this random walk on McKean graphs the *McKean process.* McKean proved the following:

McKean's Lemma [M67]: *Let γ be any member of $\Gamma(n)$. Then $p_n(\gamma)$ in (1.15) is the probability that the McKean process passes through γ at time $n-2$.*

The identification of $p_n(\gamma)$ as the probability that a random walk on McKean graphs hits the particular graph $\gamma \in \Gamma(n)$ allows us to reduce the proof of Theorem 1.1 to estimate for this process, as we shall now explain. First, note that it follows (1.12) and (1.15) that

$$f(t) - g_{(N)}(t) = \sum_{n=N+1}^{\infty} e^{-t}(1-e^{-t})^{n-1}\left(\left[\sum_{\gamma\in\Gamma(n)} p_n(\gamma)C_\gamma(F)\right] - M\right)$$

and hence

$$\|f(t) - g_{(N)}(t)\|_{L^1} = \sum_{n=N+1}^{\infty} e^{-t}(1-e^{-t})^{n-1}\left\|\left[\sum_{\gamma\in\Gamma(n)} p_n(\gamma)C_\gamma(F)\right] - M\right\|_{L^1} .$$

Therefore, if we can show

$$\left\|\left[\sum_{\gamma\in\Gamma(n)} p_n(\gamma)C_\gamma(F)\right] - M\right\|_{L^1} \le Ab^{\ln n} \tag{1.17}$$

we obtain the bound claimed in Theorem 1.1 since

$$\sum_{n=N+1}^{\infty} e^{-t}(1-e^{-t})^{n-1} = (1-e^{-t})^N .$$

Now the fact that will make it possible to prove (1.17), and hence to estimate the effects of truncation of the Wild sum, is that for n large most of the graphs γ in $\Gamma(n)$ are such that

$$C_\gamma(F) \approx M ,$$

and this holds because of an analog of the central limit theorem for the Wild convolution. To see why this might be so, let F be any probability density on $\mathbb{R}^3$ with zero mean and unit variance. Let $F^{[1]} = F \circ F$ and recursively define

$$F^{[k]} = F^{[k-1]} \circ F^{[k-1]}$$

Then, as can be proved, for example, by using the entropy production inequality of Carlen and Carvalho [CC92], the entropy strictly increases at each stage, tending to the entropy of M, and hence,

$$\lim_{k\to\infty} F^{[k]} = M .$$

Thus, for favorable arrangements of the parentheses, such as those in $C_\gamma(F) = F^{[k]}$, we will have $C_\gamma(F) \approx M$. Note that when $C_\gamma(F) = F^{[k]}$, γ is a completely balanced graph with all of its leaves at the same depth, namely k.

However, for unfavorable arrangements, we will not have $C_\gamma(F) \approx M$. For example, if we rearrange the parentheses on $F^{[k]}$ to produce $G \circ F$ where G is any $2^k - 1$ fold convolution of F, as in (1.14). Then $G \circ F$ will not be close to M, no matter how big k is, unless F itself is already close.

In the example just discussed, there is one leaf on the corresponding graph which has a depth of one. We may divide $\Gamma(n)$ into two classes of graphs: "good" graphs and "bad" graphs. The good ones are those in which the depths of all of the leaves are comparable to $\ln_2(n)$, and the bad ones are those in which there are leaves of small depth. The convolution in (1.13) corresponds to a good graph – the best sort in fact – and (1.14) to a bad graph – the worst sort in fact.

We shall show that when n is large, there will be many more "good" graphs γ in $\Gamma(n)$ such that

$$\|C_\gamma(F) - M\|_{L^1} << \|F - M\|_{L^1}$$

than there are "bad" graphs for which

$$\|C_\gamma(F) - M\|_{L^1}$$

is not small.

To prove this we need a precise measure of good and bad. The above discussion suggests that the deeper *the most shallow* of the leaves of a graph is, the better the graph is. We shall measure how good or bad a graph is through the following function that is small when all of the leaves are deep:

Definition: *Fix any constant* $0 < c < 1$. *Then for all* n *and all* $\gamma \in \Gamma(n)$ *define*

$$W(\gamma) = \sum_{j=1}^{n} \left(\frac{c}{2}\right)^{d(j)}$$

where $d(j)$ *denotes the depth of the* j*th leaf of* γ *and define*

$$T(n) = \sum_{\gamma \in \Gamma(n)} p_n(\gamma) W(\gamma)$$

Note that if $n = 2^k$, and γ is the "best possible" graph; i.e., γ is perfectly balanced so that each of the leaves of γ has depth k, then

$$W(\gamma) = 2^k \left(\frac{c}{2}\right)^k = c^k$$

which is small for k large since $c < 1$. On the other hand, if γ has even a single leaf of depth 2 for example, then $W(\gamma) > (c/2)^2$ no matter how big k is. Thus, $W(\gamma)$ does provide a quantitative measure of how deep the most shallow leaf of a graph γ is. The fact that it is just the right measure of "how good" a graph is will become clear below.

First however, note that the quantity $T(n)$ defined above satisfies

$$T(n) = \mathrm{E}W(\gamma(n)) \ .$$

That is, $T(n)$ is the expected value of W at the graph $\gamma(n)$ produced at the nth step in McKeans random walk. The extent to which good graphs dominate is then quantitatively estimated in the following lemma:

Lemma 1.2 *There is a finite constant A such that*

$$T(n) \leq Ab^{\ln n} \qquad \text{for all} \quad n \geq 2 \tag{1.18}$$

where b is any number satisfying $b > e^{c-1}$. Furthermore, the constant A depends only on the choice of b, and is explicitly computable.

The next lemma explains where the function W comes from:

Lemma 1.3 *Let ϕ be any functional on probability densities such that for some constant $c < 1$*

$$\phi(F \circ G) \leq \frac{c}{2}\Big(\phi(F) + \phi(G)\Big) \tag{1.19}$$

for all probability densities F and G with mean zero and unit variance. Then

$$\phi\left(C_\gamma(F)\right) \leq W(\gamma)\phi(F)$$

The proofs of Lemmas 1.2 and 1.3 will be given in section 2 of the paper. To apply them to the proof of Theorem 1.1, we simply need to devise an appropriate functional ϕ.

1.2 Analytic lemmas and the functional ϕ

It would have been nice if we could use $\|F - M\|_{L^1}$ but unfortunately, this does not satisfy the key requirement (1.19). What we need is a *convex* functional ϕ satisfying (1.19) and such that

$$\|G - M\|_{L^1}^q \leq C\phi(G) \tag{1.20}$$

for some $0 < q, C < \infty$, and all probability densities G of zero mean and unit variance, or at least a sufficiently large subset of all such probability densities.

Then, since ϕ is convex, application of Lemma 1.3 yields

$$\begin{aligned}\phi\Big(\sum_{\gamma\in\Gamma(n)} p_n(\gamma)C_\gamma(F)\Big) \le \\ &\sum_{\gamma\in\Gamma(n)} p_n(\gamma)\phi\left(C_\gamma(F)\right) \le \\ &\sum_{\gamma\in\Gamma(n)} p_n(\gamma)W(\gamma)\phi\left(F\right) = \\ &T(n)\phi\left(F\right)\ .\end{aligned}$$

Therefore, if we had a convex functional ϕ satisfying (1.19) and (1.20), then we would have

$$\Big\|\Big[\sum_{\gamma\in\Gamma(n)} p_n(\gamma)C_\gamma(F)\Big] - M\Big\|_{L^1}^q \le CT(n)\phi(F)$$

and hence by Lemma 1.2 we would have (1.17) provided $\phi(F)$ is finite.

It now remains to devise the appropriate functional. Our starting point is a family of norms recently introduced in [GTW95].

Definition: *Let $\alpha > 0$, and then for any integrable function H on $\mathbb{R}^3$ define*

$$|\!|\!|H|\!|\!|_\alpha = \sup_{\xi\in\mathbb{R}^3} \frac{|\widehat{H}(\xi)|}{|\xi|^{2+\alpha}} \tag{1.21}$$

where $\widehat{H}(\xi)$ denotes the Fourier transform of $H(v)$.

Here we shall take $\alpha = \delta/(1+\delta)$, where $\delta > 0$ is as in Theorem 1.1, and we shall show in section 3 of this paper that for this choice of α, there is a constant $c_\alpha < 1$ so that

$$|\!|\!|F\circ G - M|\!|\!|_\alpha \le \frac{c_\alpha}{2}\left(|\!|\!|F - M|\!|\!|_\alpha + |\!|\!|G - M|\!|\!|_\alpha\right) \tag{1.22}$$

for all probability densities F and G of mean zero and unit variance. (Actually, we need a slightly more general form, for technical reasons unfit for discussion in the introduction; see Lemma 3.1.) The fact that we can readily use the Fourier transform is another feature of Maxwellian molecules that is to our advantage here, but in using it, we further tie our analysis to this case.

Now the $|\!|\!|\cdot|\!|\!|_\alpha$ norm is weaker than the L^1 norm, so that (1.20) would not hold in general if we took $\phi(F) = |\!|\!|F - M|\!|\!|_\alpha$. However, if F belongs to an appropriate Sobolev space H^k, as in Theorem 1.1, then we can use interpolation inequalities to bound the L^1 norm of $F - M$ in terms of the $|\!|\!|\cdot|\!|\!|_\alpha$ norm and the Sobolev norm of $F - M$ using the fact that F and M are both zero mean, unit variance probability densities. The following lemma is proved in section 4 of this paper:

Lemma 1.4 *Let f be any integrable function on $\mathbb{R}^3$ such that*

$$\int_{\mathbb{R}^3} |f(v)|(1+|v|^{2+\delta})\mathrm{d}v \leq B < \infty$$

and

$$\|f\|_{\mathrm{H}^{2+\delta}} \leq C < \infty$$

for any $\delta > 0$. Then there is a constant $D < \infty$ depending only on B, C and δ so that

$$\|f\|_{L^1}^q \leq D \|\!|f|\!\|_{\delta/(1+\delta)} \ .$$

Moreover, for any $k > 0$, if F is a probability density of zero mean and unit variance such that

$$\|F\|_{\mathrm{H}^k} \leq B < \infty$$

then there is a constant $\widetilde{B} < \infty$ depending only on B so that

$$\|C_\gamma(F)\|_{\mathrm{H}^k} \leq \widetilde{B} < \infty$$

for all $\gamma \in \Gamma(n)$ and all n.

The second part of the Lemma is what makes the first part uniformly applicable in our context, at least for initial data F belonging to $\mathrm{H}^{2+\delta}$. This is why the hypothesis that $F \in \mathrm{H}^{2+\delta}$ is present in Theorem 1.1. Both parts of the lemma are closely patterned on related results in [CGT97].

There remains one difficulty in the way of proving Theorem 1.1. The difficulty is that unless F has all of its moments of order 2 equal to those of M; i.e., unless its covariance matrix is the identity then

$$\|\!|F - M|\!\|_\alpha$$

will not be finite for any $\alpha > 0$. To see this, note that under the condition

$$\int_{\mathbb{R}^3} |v|^{2+\delta} F(v)\mathrm{d}v \leq B$$

implied by the hypotheses of Theorem 1.1,

$$\widehat{F}(\xi) = 1 - \frac{1}{2}\sum_{i,j} c_{i,j}\xi_i\xi_j + O(|\xi|^{2+\delta/(1+\delta)}) \tag{1.23}$$

where $c_{i,j}$ is the covariance matrix of F, while

$$\widehat{M}(\xi) = 1 - \frac{1}{2}|\xi|^2 + O(|\xi|^3) \ . \tag{1.24}$$

For initial data F that happens to have the identity as its covariance matrix,

$$\widehat{F}(\xi) - \widehat{M}(\xi) = O(|\xi|^{2+\delta/(1+\delta)})$$

and so for any $\alpha \leq \delta/(1+\delta)$ we can indeed take $\phi(F) = \|F - M\|_\alpha$ and prove Theorem 1.1 in this case just using the above ingredients. In fact, it is exactly the bound (1.23) that determines our choice $\alpha = \delta/(1+\delta)$: This is the largest value of α for which the norms $\|\cdot\|_\alpha$ that we work with are sure to be finite, even in the case of radial initial data. The bound in (1.23) is fairly standard, but a short proof is included in section 4 for completeness.

To handle the general case, we pick the same value of α, but we also need to keep track of the effects of Wild convolution on covariances. Toward this end, define

$$p_{i,j}(F) = \int_{\mathbb{R}^3} \left(v_i v_j - \frac{1}{3}\delta_{i,j}|v|^2 \right) F(v)\mathrm{d}v \ . \tag{1.25}$$

The following lemma will be proved in section 3:

Lemma 1.5 *For all probability densities F and G of mean zero and unit variance,*

$$p_{i,j}(F \circ G) = \frac{a}{2}\left(p_{i,j}(F) + p_{i,j}(G)\right) \tag{1.26}$$

where

$$a = \frac{3}{4}\int_0^\pi B\left(\cos(\theta)\right)\cos^2\theta \sin\theta d\theta + \frac{1}{4} < 1 \tag{1.27}$$

Notice that (1.26) is an equality. The result is closely related to results of Inkenberry and Truesdell [IT56] on the evolution of moments for Maxwellian molecules.

Now, given F, define P_F to be the function whose Fourier transform $\widehat{P}_F$ is given by

$$\widehat{P}_F(\xi) = -\frac{1}{2}\sum_{i,j} p_{i,j}(F)\xi_i\xi_j\psi(|\xi|) \tag{1.28}$$

where $\psi(s)$ is a smooth, monotone decreasing function of s so that for some $0 < L_1 < L_2 < 1$, $\psi(s) = 1$ for $s \leq L_1$ and $\psi(s) = 0$ for $s \geq L_2$. Note that by (1.23), (1.24) and (1.28),

$$\widehat{F}(\xi) - \widehat{P}_F(\xi) - \widehat{M}(\xi) = O(|\xi|^{2+\delta/(1+\delta)}) \tag{1.29}$$

and hence $\| (F - P_F) - M \|_\alpha$ is finite for $\alpha = \delta/(1+\delta)$. Also, when

$$\left(\sum_{i,j} p_{i,j}(F)^2 \right)^{1/2} \tag{1.30}$$

is small, so is

$$\|P_F\|_{L^1} \ ,$$

and by Lemma 1.4, we will have that the quantity in (1.24) will be small when F is replaced by $C_\gamma(F)$ and γ is a good graph of high order. Hence, asymptotically, the P_F terms will become vanishingly small.

Using our results on the contractivity of the $\| \cdot \|_\alpha$ norm and Lemma 1.4, we shall therefore be able to show in section 5 that the functional $\phi_\star$ given by

$$\phi_\star(F) = \| (F - P_F) - M \|_\alpha + K \left(\sum_{i,j} p_{i,j}(F)^2 \right)^{1/2} \tag{1.31}$$

has the property (1.19) for K sufficiently large with an appropriate choice of the L_1 and L_2 figuring in the definition of ψ. Following this, we give the proof of Theorem 1.1. We conclude the paper by showing how Theorem 1.1, which gives a bound that is *algebraic in* N, implies the convergence of $f(t)$ to M at a rate that is *exponential in* t. This shows that Theorem 1.1 is qualitatively sharp, since stronger decay in N would then yield too much decay in t.

Because certain technical details complicate parts of the proof in this application, and because the method used may have application to other evolution equations with a quadratic nonlinearity, we summarize the content of lemmas 1.2 and 1.3 in the following theorem:

Theorem 1.6 *Let ϕ be any convex functional on probability densities such that for some constant $c < 1$,*

$$\phi(F \circ G) \le \frac{c}{2}\Big(\phi(F) + \phi(G) \Big)$$

for all probability densities F and G with mean zero and unit variance. Then for any b with

$$e^{c-1} < b < 1$$

there is a finite constant A depending only on b so that

$$\phi\Big(\sum_{\gamma \in \Gamma(n)} p_n(\gamma)(C_\gamma(F) \Big) \le A b^{\ln n} \qquad \text{for all} \quad n \ge 2 \ .$$

We now turn to the proofs of these results.

2. Probabilistic Lemmas

Proof of Lemma 1.3 First consider as an example $\phi\,(F \circ (F \circ F))$. Using (1.19) twice we have

$$\begin{array}{l} \phi\,(F \circ (F \circ F)) \le \\ \frac{c}{2}\phi\,(F)) + \frac{c}{2}\phi\,(F \circ F)) \le \\ \frac{c}{2}\phi\,(F)) + \frac{c}{2}(\frac{c}{2}\phi\,(F)) + \frac{c}{2}\phi\,(F))) \end{array}$$

The general case follows by a simple induction argument based on this pattern. □

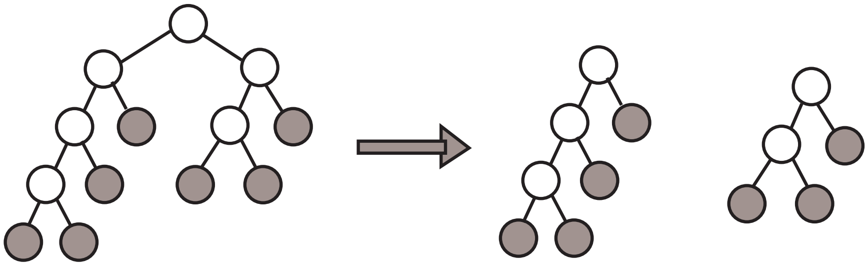
The proof of Lemma 1.2 is considerably more involved. Before giving it, we turn to some preliminary considerations.

Let $(\Gamma(n), P)$ be the discrete probability space with

$$P(\{\gamma\}) = p_n(\gamma) \tag{2.1}$$

where $\Gamma(n)$ and $p_n(\gamma)$ are defined in the introduction.

We shall analyse the contributions of the different graphs γ recursively by "splitting" $\gamma \in \Gamma(n)$ into two smaller graphs γ_ℓ and γ_r, the "left" and "right" subgraphs, by removing the root node, as shown in the diagram below:



For $n \ge 2$, let X_n denote the random variable on $(\Gamma(n), P)$ such that for any $\gamma \in \Gamma(n)$, $X_n(\gamma)$ is the number of leaves from γ that end up in γ_ℓ after the splitting. Clearly,

$$1 \le X_n \le n - 1 \tag{2.2}$$

Lemma 2.1 *For each* $n \ge 2$, X_n *is uniformly distributed on* $\{1, 2, \ldots, n-1\}$.

Our original proof of this lemma reflected the route by which it was discovered, and was more complicated. We thank Prasad Tetali for suggesting a simple inductive check.

Proof: Using McKean's lemma, we can prove this by induction. It is clearly true for $n = 2$. Now let us assume the theorem holds for all $n \le N$. If we build the graphs with the McKean process, and $X_{N+1} = k$, then either $X_N = k-1$, in which case the new γ_0 will be added on the left, or $X_N = k$ in which case the new γ_0 will be added on the right. Hence

$$\begin{aligned} P(X_{N+1} = k) =& P(X_{N+1} = k | X_N = k-1)P(X_N = k-1)+ \\ & P(X_{N+1} = k | X_N = k)P(X_N = k) \end{aligned}$$

Now, given that $X_N = k-1$, we will pass to $X_{N+1} = k$ exactly when we choose one of the $k-1$ leaves on the left as the place to attach the next γ_0. Since these sites are chosen uniformly, it follows that

$$P(X_{N+1} = k | X_N = k-1) = \frac{k-1}{N}$$

and, by similar reasoning, that

$$P(X_{N+1} = k | X_N = k) = \frac{N-k}{N} \ .$$

Then by the inductive hypothesis

$$P(X_{N+1} = k) = \frac{k-1}{N}\frac{1}{N-1} + \frac{N-k}{N}\frac{1}{N-1} = \frac{1}{N}$$

□

Proof of Lemma 1.2 Our first goal is to show that

$$T(n) = \frac{c}{n-1}\sum_{k=1}^{n-1} T(k) \tag{2.3}$$

where by convention, $T(1) = 1$. We shall prove this inductively.

Take any graph $\gamma \in \Gamma(n)$, and split it into its left and right parts, as in the proof of the previous lemma. After the splitting, for some j with $1 \le j \le n-1$, j of the leaves are in the left subgraph, γ_ℓ and $n-j$ on the right subgraph, γ_r.

Now if we think of $W(\gamma)$ as a random variable on $\Gamma(n)$, we have that $T(n)$ is its expectation; i.e.,

$$T(n) = \mathrm{E}W$$

But then, with X_n as in Lemma 2.1,

$$\begin{aligned} \mathrm{E}W = & \sum_{k=1}^{n-1} \mathrm{E}\{W | X_n = k\} P(X_n = k) = \\ & \sum_{k=1}^{n-1} \mathrm{E}\{W | X_n = k\} \frac{1}{n-1} \end{aligned}$$

by Lemma 2.1.

Moreover, as a member of the subgraph γ_ℓ or γ_r that it ends up in, each leaf is exactly one step closer to the root; i.e., its depth is reduced by one. Therefore, by the definition of W,

$$W(\gamma) = \frac{c}{2}\big(W(\gamma_\ell) + W(\gamma_r)\big) \ .$$

Next, we claim that

$$P(\gamma_\ell = \widetilde{\gamma} | X_n = k) = p_k(\widetilde{\gamma}) \ ,$$

and that

$$P(\gamma_r = \widetilde{\gamma} | X_n = k) = p_{n-k}(\widetilde{\gamma}) \ .$$

To see this, note that by the definition of the McKean process, the way in which we add the k γ_0's on the left is not influenced in any way by the way we add the $n-k$ γ_0's on the right, or the order in which the additions on the left and right occur.

Combining the last three equations, we have:

$$\mathrm{E}\{W | X_n = k\} = \frac{c}{2}\big(T(k) + T(n-k)\big)$$

Thus

$$\begin{aligned} T(n) =& \frac{c}{2}\sum_{k=1}^{n-1} \frac{1}{n-1}\big(T(k) + T(n-k)\big) = \\ & \frac{c}{n-1}\sum_{k=1}^{n-1} T(k) \end{aligned}$$

This recurrence relation (2.3) is the key to the proof. To complete the proof, let A and b be positive constants to be chosen later. We shall however already stipulate that

$$e^{-1} < b < 1 \ .$$

Define

$$a_n = Ab^{\ln n} = An^{-p} \qquad \text{where} \qquad p = |\ln b| \tag{2.4}$$

Note that by our stipulation on b, we have

$$0 < p < 1$$

and hence

$$\begin{aligned} \frac{c}{n-1}\sum_{k=1}^{n-1} a_k \leq & \frac{cA}{n-1}\left(1 + \int_1^{n-1} \frac{1}{x^p}\mathrm{d}x\right) = \\ & \frac{cA}{n-1}\left(1 + \frac{1}{1-p}\left((n-1)^{1-p} - 1\right)\right) \leq \\ & \left[\frac{c}{(1-p)}\left(\frac{n}{n-1}\right)^p\right] An^{-p} = \\ & \left[\frac{c}{(1-p)}\left(\frac{n}{n-1}\right)^p\right] a_n = \end{aligned} \tag{2.5}$$

Now, as long as $c/(1-p) < 1$, there is an N_0 so that

$$\left[\frac{c}{(1-p)}\left(\frac{n}{n-1}\right)^p\right] \le 1 \qquad \text{for all} \qquad n \ge N_0 \ . \tag{2.6}$$

But $c/(1-p) < 1$ is the same as $p < 1-c$, or $b > e^{c-1}$, which is in line with our stipulation. Therefore, fix any such b, and define

$$A = \max\{k^p T(k) \mid k = 1, 2, \dots, N_0\}$$

with N_0 as above.

By the very definition of A, we have $T(k) \le a_k$ for all $k \le N_0$. Now make the inductive hypothesis that $T(k) \le a_k$ for all $k < n$, where we may assume that $n > N_0$. Then

$$T(n) = \frac{c}{n-1}\sum_{k=1}^{n-1} T(k) = \frac{c}{n-1}\sum_{k=1}^{n-1} a_k \le a_n \tag{2.7}$$

since for $n \ge N_0$,

$$\frac{c}{n-1}\sum_{k=1}^{n-1} a_k \le a_n \tag{2.8}$$

by (2.5) and (2.6). This provides the inductive step needed to show that $T(n) \le a_n$ for all n. Moreover, it is clear from the proof that A can be explicitly computed without any trouble. □

3. Analytic Lemmas

This section contains the key analytic lemmas, beginning with a statement about Wild convolution and the $|\!|\!| \cdot |\!|\!|_\alpha$ norm.

Lemma 3.1 *Consider any four integrable functions F_1, F_2, G_1 and G_2 such that their Fourier transforms $\widehat{F}_1$ $\widehat{F}_2$, $\widehat{G}_1$ and $\widehat{G}_2$ satisfy*

$$\|\widehat{F}_i\|_\infty \ , \ \|\widehat{G}_i\|_\infty \le 1+\epsilon \qquad \text{for} \quad i = 1,2$$

where

$$(1+\epsilon)c_\alpha = c < 1$$

and

$$c_\alpha = \int |\cos(\theta/2)|^{2+\alpha} B(\theta)\cos(\theta)\mathrm{d}\theta \ .$$

Then

$$|\!|\!| F_1 \circ G_1 - F_2 \circ G_2 |\!|\!|_\alpha \le \frac{c_\alpha}{2}\left(|\!|\!| F_1 - F_2 |\!|\!|_\alpha + |\!|\!| G_1 - G_2 |\!|\!|_\alpha\right) \tag{3.1}$$

Proof We begin by using the Fourier transform of $F \circ G$, first worked out by Bobylev [B88]:

$$\widehat{F \circ G}(\xi) = \int_{S^2} \widehat{F}(\xi_+)\widehat{G}(\xi_-)B(\mathbf{n} \cdot \xi/|\xi|)\mathrm{d}\mathbf{n} \tag{3.2}$$

where

$$\xi_\pm = \frac{\xi \pm |\xi|\mathbf{n}}{2} \ .$$

Let the angle θ be given by

$$\cos(\theta) = \mathbf{n} \cdot \xi/|\xi| \ .$$

Then, as shown in, e.g., [CGT97],

$$|\xi_+| = \cos(\theta/2)|\xi| \qquad \text{and} \qquad |\xi_-| = \sin(\theta/2)|\xi| \ .$$

Now,

$$F_1 \circ G_1 - F_2 \circ G_2 = (F_1 - F_2) \circ G_1 + F_2 \circ (G_1 - G_2) \ . \tag{3.3}$$

Fourier transforming $(F_1 - F_2) \circ G_1$ one gets

$$\int_{S^2} (\widehat{F}_1(\xi_+) - \widehat{F}_2(\xi_+))\widehat{G}(\xi_-)B(\mathbf{n} \cdot \xi/|\xi|)\mathrm{d}\mathbf{n}$$

Next,

$$\begin{aligned} &\frac{|(\widehat{F}_1(\xi_+) - \widehat{F}_2(\xi_+))\widehat{G}_2(\xi_-)|}{|\xi|^{2+\alpha}} = \\ &\frac{|(\widehat{F}_1(\xi_+) - \widehat{F}_2(\xi_+))\widehat{G}_2(\xi_-)|}{|\xi_+|^{2+\alpha}\cos(\theta/2)^{-2-\alpha}} \leq \\ &\|F_1 - F_2\|_\alpha \cos(\theta/2)^{2+\alpha}\|G_2\|_\infty \leq \\ &\|F_1 - F_2\|_\alpha \cos(\theta/2)^{2+\alpha}(1+\epsilon) \end{aligned}$$

Hence, the contribution of $(F_1 - F_2) \circ G_1$ to

$$\|F_1 \circ G_1 - F_2 \circ G_2\|_\alpha$$

is bounded by

$$\|F_1 - F_2\|_\alpha(1+\epsilon)\int_{S^2} \cos(\theta/2)^{2+\alpha}B(\mathbf{n} \cdot \xi/|\xi|)\mathrm{d}\mathbf{n}$$

In the same way, one finds that the contribution of $F_2 \circ (G_1 - G_2)$ to

$$\|F_1 \circ G_1 - F_2 \circ G_2\|_\alpha$$

is bounded by

$$\|\!|F_1 - F_2\|\!|_\alpha (1+\epsilon) \int_{S^2} \sin(\theta/2)^{2+\alpha} B(\mathbf{n}\cdot\xi/|\xi|)\mathrm{d}\mathbf{n}$$

Hence with c_α given by

$$c_\alpha = \int_{S^2} \left(\sin(\theta/2)^{2+\alpha} + \cos(\theta/2)^{2+\alpha}\right) B(\mathbf{n}\cdot\xi/|\xi|)\mathrm{d}\mathbf{n}\ ,$$

which is strictly less than 1 since

$$\int_{S^2} B(\mathbf{n}\cdot\xi/|\xi|)\mathrm{d}\mathbf{n} = 1$$

□

Proof of Lemma 1.5 First, consider the case $i \neq j$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^3} v_i v_j F\circ G(v)\mathrm{d}v = \\
&\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} v_i v_j B\left(\cos(\theta)\right) F(v^\star) G(w^\star)\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}\mathbf{n} = \\
&= \int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} B\left(\cos(\theta)\right) v_i^\star v_j^\star F(v)\, G(w)\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}\mathbf{n} \\
&v_i^\star v_j^\star = \left[\frac{1}{2}\left(v_i + w_i\right) + \frac{1}{2} q n_i\right]\left[\frac{1}{2}\left(v_j + w_j\right) + \frac{1}{2} q n_j\right] \\
&= \frac{1}{4}\left(v_i v_j + w_i w_j\right) + \frac{1}{4} q^2 n_i n_j + \\
&+ \frac{1}{4}\left[v_i w_j + v_j w_i + (v_i + w_i) q n_j + \left(v_j + w_j\right) q n_i\right]
\end{aligned} \tag{3.4}$$

Since $\int_{S^2} B\left(\cos(\theta)\right)\mathrm{d}\mathbf{n} = 1$, and $B\left(\cos(\theta)\right)(\cdot)$ is an even function, the contribution corresponding to the last line of (5.1) (that is odd) is zero.

Therefore we have

$$\begin{aligned}
\int_{\mathbb{R}^3} v_i v_j F\circ G\mathrm{d}v = \frac{1}{4}\left[\int_{\mathbb{R}^3} v_i v_j F(v)\mathrm{d}v + \int_{\mathbb{R}^3} w_i w_j G(w)\mathrm{d}w\right]. \\
+ \frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} F(v) G(w) q^2 n_i n_j B\left(\cos(\theta)\right)\mathrm{d}\mathbf{n}\,\mathrm{d}v\,\mathrm{d}w
\end{aligned}$$

Now let $A(q)_{i,j}$ be the matrix defined by

$$A(q)_{i,j} = \int_{S^2} n_i n_j B\left(\cos(\theta)\right)\mathrm{d}\mathbf{n}\ .$$

Then for $i \neq j$

$$A(q)_{i,j} = \frac{3A^\star - 1}{2q^2} q_i q_j \ ,$$

where

$$A^\star = \int_0^\pi B\left(\cos(\theta)\right) \cos^2\theta \sin\theta d\theta < 1 \tag{3.5}$$

and hence

$$\begin{aligned} &\int_{\mathbb{R}^3\times\mathbb{R}^3} F(v)G(w) q^2 n_i\, n_j B\left(\cos(\theta)\right) \mathrm{d}\mathbf{n}\, \mathrm{d}v\, \mathrm{d}w = \\ &= \frac{3A^\star - 1}{2} \int_{\mathbb{R}^3\times\mathbb{R}^3} F(v)G(w)\left(v_i - w_i\right)(v_j - w_j) \mathrm{d}v\, \mathrm{d}w = \\ &= \frac{3A^\star - 1}{2} \left[\int_{\mathbb{R}^3} v_i v_j F(v)\mathrm{d}v + \int_{\mathbb{R}^3} w_i w_j G(w)\mathrm{d}w\right] \end{aligned}$$

Finally, for $i = j$,

$$\int_{\mathbb{R}^3} v_i v_j F \circ G \mathrm{d}v = \frac{a}{2} \int_{\mathbb{R}^3} v_i v_j F(v)\mathrm{d}v + \frac{a}{2} \int_{\mathbb{R}^3} v_i v_j G(v)\mathrm{d}v \tag{3.6}$$

with

$$a = \frac{3A^\star + 1}{4} < 1\,. \tag{3.7}$$

If now $i = j$

$$A_{ij}(q) = \frac{3A^\star - 1}{2q^2}\, q_i^2 + \frac{1 - A^\star}{2}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^3} v_i^2 F \circ G = \frac{1}{4}\left[\int_{\mathbb{R}^3} v_i^2 F(v)\mathrm{d}v + \int_{\mathbb{R}^3} v_i^2 G(v)\mathrm{d}v\right] \\ &+ \frac{1}{4}\left[\frac{3A^\star - 1}{2} \int_{\mathbb{R}^3} q^2 F(v)G(w)\mathrm{d}v\, \mathrm{d}w + \frac{1 - A^\star}{2} \int_{\mathbb{R}^3} q^2 F(v)G(w)\mathrm{d}v\, \mathrm{d}w\right] \\ &= \frac{a}{2} \int_{\mathbb{R}^3} v_i^2 F(v)\mathrm{d}v + \frac{a}{2} \int_{\mathbb{R}^3} v_i^2 G(v)\mathrm{d}v + \frac{1 - A^\star}{8} \left[\int_{\mathbb{R}^3} v^2 F + \int_{\mathbb{R}^3} v^2 G\right] \end{aligned}$$

Finally, for $i \neq j$,

$$\begin{aligned} &\int_{\mathbb{R}^3} \left(v_i^2 - \frac{1}{3}v^2\right) F \circ G = \frac{a}{2} \int_{\mathbb{R}^3} v_i^2 F(v)\mathrm{d}v + \frac{a}{2} \int_{\mathbb{R}^3} v_i^2 G(v)\mathrm{d}v \\ &+ \frac{1 - A^\star}{8} \left[\int_{\mathbb{R}^3} v^2 F + \int_{\mathbb{R}^3} v^2 G\right] - \frac{1}{6} \left[\int_{\mathbb{R}^3} v^2 F + \int_{\mathbb{R}^3} v^2 G\right] \\ &= \frac{a}{2} \int_{\mathbb{R}^3} \left(v^2 - \frac{1}{3}v^2\right) F(v)\mathrm{d}v + \frac{a}{2} \int_{\mathbb{R}^3} \left(v_i^2 - \frac{1}{3}v^2\right) G(v)\mathrm{d}v \end{aligned} \tag{3.8}$$

Combining (3.6), (3.7) and (3.8), the lemma is proved. □

4. Interpolation Estimates

Here we recall some interpolation inequalities that allow us to pass from the $\|\!| \cdot |\!\|_\alpha$ norm to the $\|\cdot\|_{L^1}$ norm. The first two of these are essentially specializations of lemmas in [CGT97].

Lemma 4.1 *Let $0 < \alpha, r < 1$ and $\beta > 0$ be given. Then for any function f on $\mathbb{R}^3$,*

$$\|f\|_{L^2}^2 \le C(r,\beta) \|\!| f |\!\|_\alpha^{2(1-r)} \left(\|f\|_{\mathrm{H}^M}^{2r} + \|f\|_{\mathrm{H}^N}^{2r} \right)$$

with

$$M = \frac{(2+\alpha)(2-r)}{2r} \qquad\qquad N = M + r(1-r)(3+\beta) \tag{4.1}$$

$$C(r,\beta) = \left(|B(3)|(1+3/\beta) \right)^{1-r}$$

and where $|B(3)|$ denotes $4\pi/3$, the volume of the unit ball in $\mathbb{R}^3$.

Proof: For any $p > 0$, and any r with $0 < r < 1$,

$$\|f\|_{L^2}^2 = \int_{\mathbb{R}^3} |\widehat{f}(\xi)|^2 \mathrm{d}\xi =$$

$$\int_{\mathbb{R}^3} \frac{|\widehat{f}(\xi)|^{2-2r}}{|\xi|^{(2+\alpha)(2-2r)}} |\widehat{f}(\xi)|^{2r} |\xi|^{(2+\alpha)(2-2r)} (1+|\xi|^p)^r (1+|\xi|^p)^{-r} \mathrm{d}\xi \le$$

$$\|\!| f |\!\|_\alpha^{2-2r} \int_{\mathbb{R}^3} |\widehat{f}(\xi)|^{2r} |\xi|^{(2+\alpha)(2-2r)} (1+|\xi|^p)^r (1+|\xi|^p)^{-r} \mathrm{d}\xi \le$$

$$\|\!| f |\!\|_\alpha^{2-2r} \left(\int_{\mathbb{R}^3} |\widehat{f}(\xi)|^2 |\xi|^{((2+\alpha)(2-2r))/r} (1+|\xi|^p) \mathrm{d}\xi \right)^r \times$$

$$\left(\int_{\mathbb{R}^3} (1+|\xi|^p)^{-r/(1-r)} \mathrm{d}^3\xi \right)^{1-r} .$$

For the last integral to converge, we require that $p > (1-r)3/r$, so let $\beta > 0$, and put $p = (1-r)(3+\beta)/r$. Then the integral is

$$\int_{\mathbb{R}^3} (1+|\xi|^{(1-r)(3+\beta)/r})^{-r/(1-r)} \mathrm{d}\xi \le$$

$$|B(3)| + 3|B(3)| \int_1^\infty r^{-(1+\beta)} \mathrm{d}r = |B(3)|(1+3/\beta) .$$

□

The next inequality shows that control of the second moments and control on the L^2 norm together control the L^1 norm.

Lemma 4.2 *Let f be an integrable function on $\mathrm{I\!R}^3$. Then*

$$\int_{\mathrm{I\!R}^3} |f(v)|\mathrm{d}v \le C\Big(\int_{\mathrm{I\!R}^3} |f(v)|^2\mathrm{d}v\Big)^{\frac{2}{7}}\Big(\int_{\mathrm{I\!R}^3} |v|^2|f(v)|\mathrm{d}v\Big)^{\frac{3}{7}}$$

where

$$C = \left[\left(\frac{3}{4}\right)^{4/7} + \left(\frac{4}{3}\right)^{3/7}\right]|B(3)|^{2/7}$$

and again, $|B(3)|$ denotes $4\pi/3$, the volume of the unit ball in $\mathrm{I\!R}^3$.

Proof: Let $R > 0$ be chosen. Then

$$\int_{\mathrm{I\!R}^3} |f(v)|\mathrm{d}v = \int_{|v|\le R} |f(v)|\mathrm{d}v + \int_{|v|\ge R} |f(v)|\mathrm{d}v \le$$

$$\big(|B(3)|R^3\big)^{1/2}\|f\|_{L^2} + R^{-2k}\int_{\mathrm{I\!R}^3} |v|^{2k}f(v)\mathrm{d}v \ .$$

Optimizing in R now yields the result. $\square$

Now consider any function f on $\mathrm{I\!R}^3$ such that

$$\int_{\mathrm{I\!R}^3} |f(v)|(1+|v|^2)\mathrm{d}v \le B < \infty \ .$$

Then for a constant C depending only on B,

$$\|f\|_{L^1}^2 \le C\|f\|_{L^2}^{4/7}$$

by Lemma 4.2. Now fix any $\delta > 0$. Fix $\beta = 1/2$, and note that for any α satisfying $0 < \alpha < 1$, we can choose $r > 0$ so that the constants M and N in (4.1) satisfy

$$2 < M < N \le 2 + \delta \ .$$

Next, since $\|f\|_\infty$ can be bounded in terms of $\|f\|_{\mathrm{H}^k}$ for any $k \ge 2$, $\|f\|_{L^2}^2$ can be bounded in tersm of $\|f\|_{\mathrm{H}^N}$ and B. Hence, by a trivial interpolation, $\|f\|_{\mathrm{H}^M}$ can be bounded in terms of $\|f\|_{\mathrm{H}^N}$ and B, and we have the following result:

Lemma 4.3 *Let f be a function on $\mathrm{I\!R}^3$ such that*

$$\int_{\mathrm{I\!R}^3} |f(v)|(1+|v|^2)\mathrm{d}v \le B < \infty \ .$$

Suppose further that for some $\delta > 0$,

$$\|f\|_{\mathrm{H}^{2+\delta}} \leq C \ .$$

Then there are constants $D < \infty$ and $q > \infty$ depending only on B, C and δ so that

$$\|f\|_{L^1}^q \leq D\|\!|f|\!\|_\alpha \ .$$

Proof of Lemma 1.4 All but the final part of Lemma 1.4 is contained in Lemma 4.3, and the final part, giving uniform bounds on $\|C_\gamma(F)\|_{\mathrm{H}^{2+\delta}}$ in terms of $\|F\|_{\mathrm{H}^{2+\delta}}$ is given in [CGT97]. We remark that the bound we have assumed on the $2+\delta$th moment of the initial data will also propagate uniformly; an inequality of Elmroth [E82] provides this. However, the only role of this condition is to ensure that $\phi_\star(F) < \infty$, so we do not need to concern ourselves with propagation of moments.

Finally, we prove a lemma that will give us the bound (1.23). This is a simple variant of the Riemann-Lebesgue argument, and is well known in many circles, but we include it for completeness since it is so short. To get (1.23) from the following lemma, consider $g(v) = v_iv_jF(v)$ and $\eta = 0$.

Lemma 4.4 *There is a universal constant C such that for any function g on $\mathbb{R}^3$ such that*

$$\int_{\mathbb{R}^3} |g(v)|(1+|v|^\delta)\mathrm{d}v \leq B < \infty \ .$$

Then for any ξ and η,

$$|\widehat{g}(\xi) - \widehat{g}(\eta)| \leq C|\xi - \eta|^{\delta/(1+\delta)} \ .$$

Proof of Lemma 4.4 For any $R > 0$,

$$\begin{aligned}
&|\widehat{g}(\xi) - \widehat{g}(\eta)| \leq \\
&\int_{\mathbb{R}^3} |e^{-i(\xi-\eta)\cdot v} - 1||g(v)|\mathrm{d}v \leq \\
&\int_{|v|\leq R} |e^{-i(\xi-\eta)\cdot v} - 1||g(v)|\mathrm{d}v + \int_{|v|\geq R} |e^{-i(\xi-\eta)\cdot v} - 1||g(v)|\mathrm{d}v \leq \\
&|\xi-\eta|R\int_{\mathbb{R}^3} |g(v)|\mathrm{d}v + 2R^{-\delta}\int_{\mathbb{R}^3} |g(v)||v|^\delta\mathrm{d}v \leq \\
&B\left(|\xi-\eta|R + 2R^{-\delta}\right) \ .
\end{aligned}$$

Optimizing in R now yields the result. $\square$.

5. Proof of the Main Theorem

Recall the definition (1.28):

$$\widehat{P}(F)(\xi) = -\frac{1}{2}\sum_{i,j} p_{ij}(F)\xi_i\xi_j\psi(|\xi|) \tag{5.1}$$

where ψ is a smooth function of ξ, such that $\psi(|\xi|) = 1$ for $|\xi| \leq L_1$ and $\psi(|\xi|) = 0$ for $|\xi| \geq L_2$, $0 < L_1 < L_2$.

Our first goal is to show that for any $\epsilon > 0$, with L_2 chosen sufficiently small, we can achieve

$$\|\widehat{P}_F\|_\infty \leq \epsilon$$

so that Lemma 3.1 will be applicable to $F - P_F$.

With this purpose in mind, observe that

$$|\widehat{P}(F)(\xi)| \leq \frac{1}{2}\sum_{i,j} |p_{ij}(F)|L_2^2$$

and so

$$\|\widehat{F} - \widehat{P}(F)\|_\infty \leq 1 + \frac{1}{2}\sum_{i,j} |p_{ij}(F)|L_2^2 \tag{5.2}$$

which is all that we need with an appropriate choice of L_2. We are now in a position to apply Lemma 3.1.

Our second goal is to show that the functional $\phi_\star$ defined in (1.31) has the key property (1.19). The following lemma eases the way:

Lemma 5.1 *There is a finite constant D so that for any two probability densities F and G with zero mean and unit variance,*

$$\|P_{F\circ G} - F\circ P_G - G\circ P_F + P_F\circ P_G\|_\alpha \leq D\left(\max_{i,j} P_{i,j}(F) + \max_{i,j} P_{i,j}(G)\right)$$

Proof: We first consider ξ with $|\xi| \leq L_1$. Since $|\xi_\pm| \leq |\xi|$, it follows from the definitions that

$$\widehat{P}(F)(\xi_\pm) = -\frac{1}{2}\sum_{i,j} p_{ij}(F)(\xi_\pm)_i(\xi_\pm)_j$$

for such ξ, with the same holding for P_G. Also, since F and G are proability densities of zero mean and unit variance, there is a universal constant C so that

$$|\widehat{F}(\xi) - 1| \ , \ |\widehat{G}(\xi) - 1| \le C|\xi|^2 \ .$$

Hence, from (3.2), for $|\xi| \le L_1$,

$$\widehat{F \circ P_G} = \int_{S^2} \widehat{F}(\xi_+)\widehat{P}_G(\xi_-)B(\mathbf{n}\cdot\xi/|\xi|)\mathrm{d}\mathbf{n} =$$
$$\int_{S^2} -\frac{1}{2}\sum_{i,j} p_{ij}(G)(\xi_-)_i(\xi_-)_j B(\mathbf{n}\cdot\xi/|\xi|)\mathrm{d}\mathbf{n} + \int_{S^2}\left(\widehat{F}(\xi_+) - 1\right)\widehat{P}_G(\xi_-)B(\mathbf{n}\cdot\xi/|\xi|)\mathrm{d}\mathbf{n} \ .$$

Now by the same computations on the spherical integral that were done in the proof of Lemma 1.5,

$$\int_{S^2} -\frac{1}{2}\sum_{i,j} p_{ij}(G)(\xi_-)_i(\xi_-)_j B(\mathbf{n}\cdot\xi/|\xi|)\mathrm{d}\mathbf{n} = \frac{a}{2}\left(-\frac{1}{2}\sum_{i,j} p_{ij}(G)\xi_i\xi_j\right)$$

where a is the constant in Lemma 1.5. This, together with a similar computation for $P_F \circ G$, and Lemma 1.5, yield the result that

$$|\widehat{P_{F\circ G}} - \widehat{F \circ P_G} - \widehat{P_F \circ G}| \le$$
$$\int_{S^2}\left|\widehat{F}(\xi_+) - 1\right|\widehat{P}_G(\xi_-)B(\mathbf{n}\cdot\xi/|\xi|)\mathrm{d}\mathbf{n} + \int_{S^2}\left|\widehat{G}(\xi_-) - 1\right|\widehat{P}_F(\xi_+)B(\mathbf{n}\cdot\xi/|\xi|)\mathrm{d}\mathbf{n} \le$$
$$\left(\left(\sum_{i,j} p_{i,j}^2(F)\right)^{1/2} + \left(\sum_{i,j} p_{i,j}^2(G)\right)^{1/2}\right) C|\xi|^4$$

for $|\xi| \le L_1$.

Even more easily, one obtains a bound of the same sort for $|\widehat{P_F \circ P_G}|$. Thus the terms that are second order in ξ all cancel out, and there is no difficulty in dividing by $|\xi|^{2+\alpha}$ for $|\xi| \le L_1$. But there is clearly no problem in doing so for $|\xi| \ge L_1$, and since every term in the quantity we are estimating involves at least one of $p_{i,j}(F)$ or $p_{i,j}(G)$, we obtain the desired bound for $|\xi| \ge L_1$ simply by using the Schwarz inequality, without being concerned with cancelation. This proves the lemma. □

Proof of Theorem 1.1: Pick $\alpha = \delta/(1+\delta)$ so that $\phi_\star(F)$ will be finite, by (1.23). Observe that by lemmas 1.5, 3.1 and 5.1, and with $\phi_\star$ defined as in (1.31), with this choice of α,

$$\phi_\star(F \circ G) \le$$
$$\frac{c_\alpha}{2}\left(|||F - P_F - M|||_\alpha + |||G - P_G - M|||_\alpha\right) +$$
$$\left(D + K\frac{a}{2}\right)\left(\max_{i,j}|p_{i,j}(F)| + \max_{i,j}|p_{i,j}|(G)\right) \ .$$

Now pick b with $a < b < 1$. For any such b, we can choose K so that

$$D + K\frac{a}{2} = K\frac{b}{2} \ .$$

Hence, with $c = \max\{c_\alpha, b\}$, we have

$$\phi_\star(F \circ G) \le \frac{c}{2}\left(\phi_\star(F) + \phi_\star(G)\right) \ .$$

Hence $\phi_\star$ has the property (1.19). Moreover, since the map $F \mapsto p_{i,j}(F)$ is linear, $\phi_\star$ is clearly convex. Then by lemmas 1.2 and 1.3, for any b with $e^{c-1} < b < 1$, there is a finite constant A such that

$$\sum_{\gamma\in\Gamma(n)} p_n(\gamma)\phi_\star\left(C_\gamma(F)\right) \le Ab^{\ln n} \ .$$

Finally, Theorem 1.1 follows directly from this and Lemma 1.4. □

We conclude the paper by briefly discussing the sharpness of the bounds. As noted earlier

$$b^{\ln n} = n^{-|\ln b|}$$

and hence the bounds provided by Theorem 1.1 are algebraic in n, the order of truncation of the Wild sum. However, the bound is exponential in $\ln_2 n$, which measure the depth of a typical leaf of a good graph γ in $\Gamma(n)$. That is, $\ln_2 n$ is roughly the number of "generations" separating a typical leaf from the root in a good McKean graph, and hence measures the "generation" of the most recent collision in the corresponding collision history. A bound that gives exponential decrease in terms of the typical generation is the best one could hope for, as one easily sees reflecting upon the proof. Moreover, it suffices to show that a solution $f(t)$ of (1.5) with initial data F satisfying the hypotheses of Theorem 1.1 converges to M exponentially fast in t. To see this, note that by Theorem 1.1, for any N,

$$\begin{aligned}
&\|f(t) - M\|_{L^1} \le \|f(t) - g_N(t)\|_{L^1} + \|g_N(t) - M\|_{L^1} \le \\
&Ab^{\ln N} + \sum_{n=0}^{N-1} e^{-t}(1-e^{-t})^{n-1}\left\|\left[\sum_{\gamma\in\Gamma(n)} p_n(\gamma)C_\gamma(F)\right] - M\right\|_{L^1} \le \\
&Ab^{\ln N} + Ne^{-t}2
\end{aligned}$$

Now choose $N = e^{t/2}$ and obtain

$$\|f(t) - M\|_{L^1} \le Ae^{-|\ln b|t/2} + 2e^{-t/2} \ .$$

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