

# Approximate solution of the Cahn-Hilliard equation via corrections to the Mullins-Sekerka motion.

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**Abstract** We develop an alternative method to matched asymptotic expansions for the construction of approximate solutions of the Cahn-Hilliard equation suitable for the study of its sharp interface limit. The method is based on the Hilbert expansion used in kinetic theory. Besides its relative simplicity, it leads to calculable higher order corrections to the interface motion.

## 1. Introduction

### 1.1 The Cahn–Hilliard equation and phase segregation

The purpose of this paper is to present a method for constructing approximate solutions to a class of evolution equations typified by the Cahn–Hilliard equation. The method produces solutions suitable for studying the sharp interface limit, and for studying higher order corrections to the sharp interface limit. The method itself is based on the Hilbert expansion used in kinetic theory [5]. The work of Caflisch [4] on constructing solutions of the Boltzmann equation from solutions of the Euler equations can be considered as a paradigm for this sort of investigation. The sharp interface limit of the Cahn–Hilliard equation itself has been rigorously investigated by Alikakos, Bates and Chen [1], following the original heuristic analysis of Pego [13]. Both [13] and [1] are based on matched asymptotic expansions. We aim to show that the Hilbert expansion approach has advantages in the presence of the non–locality inherent in this class of problems, and that in any case, it provides a means to calculate higher order corrections to the sharp interface limit. We begin by recalling some background.

Let  $\Omega$  be a compact domain in  $\mathbb{R}^2$ . The restriction to two dimensions is for simplicity only; we seek to explain the main ideas in the simplest interesting setting. Let  $m$  be an integrable function on  $\Omega$ . We think of  $m(x, t)$  as representing the value of a conserved “order parameter” at  $x$  in  $\Omega$  at time  $t$ . The order parameter is conserved in the sense that  $\int_{\Omega} m(x, t) dx$  is independent of  $t$ . Therefore, the evolution equation for  $m$  can be written in the form

$$\frac{\partial}{\partial t} m(x, t) = \nabla \cdot \vec{J}(x, t)$$

where the *current*  $\vec{J}$  is orthogonal to the normal at the boundary of  $\Omega$ . In the class of equations to be considered, the current will have the form

$$\vec{J}(x, t) = \sigma(m(x)) \nabla \mu(x)$$

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\* Work partially supported by U.S. National Science Foundation grant DMS 92–07703

\*\* Work partially supported by E.U. grant CHRX–CT93–0411 and PRAXIS XX!/BPD/16400/98

\*\*\* Work partially supported by the CNR–GNFM, MURST Cofin 2003/2004

where the  $\sigma(m)$  is the *mobility* and  $\mu(x)$  is the *chemical potential* of  $x$ . The mobility is positive, so that the conserved order parameter  $m$  “flows” in the direction of increasing chemical potential.

Finally, the chemical potential is the  $L^2(\Omega)$  Frechet derivative of a *free energy functional*  $\mathcal{F}$ :

$$\mu(x) = \frac{\delta \mathcal{F}}{\delta m}(x) .$$

The simplest and most familiar example is known as the Cahn–Hilliard equation. It results from the choices  $\sigma(m) = 1$ ; i.e., constant mobility, and \*

$$\mathcal{F}(m) = \frac{1}{2} \int_{\Omega} |\nabla m(x)|^2 dx + \frac{1}{4} \int_{\Omega} (m^2(x) - 1)^2 dx .$$

This leads to

$$\frac{\partial}{\partial t} m(x, t) = \Delta (-\Delta m(x, t) + f(m(x, t))) ,$$

where

$$f(m) = m^3 - m . \tag{1.1}$$

If  $m(x, t)$  is a solution of this equation, then

$$\frac{d}{dt} \mathcal{F}(m(\cdot, t)) = - \int_{\Omega} \frac{1}{\sigma} |\vec{J}(x, t)|^2 dx ,$$

so that the evolution decreases the free energy. Also clearly, the minimizers of the free energy are the constant functions  $m(x) = \pm 1$ . These minimizers represent the “pure phases” of the system. However, unless the initial data  $m_0$  happens to satisfy  $\int_{\Omega} m_0(x) dx = \pm |\Omega|$ , these “pure phases” cannot be reached because of the conservation law. Instead, what will eventually be produced is a region in which  $m(x) \approx +1$ , with  $m(x) \approx -1$  in its complement, and with a smooth transition across its boundary. This is referred to a *phase segregation*, and the boundary is the *interface* between the two phases. If we “stand far enough back” from  $\Omega$ , all we see is the interface, and we do not see any structure across the interface – the structure now being on an invisibly small scale. The evolution of  $m$  under the Cahn Hilliard equation, or another such evolution equation of this type, drives an evolution of the interface, and we wish to determine how it evolves. To see any evolution of the interface, one must wait a long time. More specifically, let  $\lambda$  be a small parameter, and introduce new variables  $\tau$  and  $\xi$  through

$$\tau = \lambda^3 t \quad \text{and} \quad \xi = \lambda x .$$

Then of course

$$\frac{\partial}{\partial t} = \lambda^3 \frac{\partial}{\partial \tau} \quad \text{and} \quad \frac{\partial}{\partial x} = \lambda \frac{\partial}{\partial \xi} .$$

Hence if  $m(x, t)$  is a solution of the Cahn–Hilliard equation, and we define  $m^\lambda(\xi, \tau)$  by  $m^\lambda(\xi, \tau) = m(x(\xi), t(\tau))$ , we obtain

$$\frac{\partial}{\partial \tau} m^\lambda(\xi, \tau) = \Delta_\xi \left( -\lambda \Delta_\xi m^\lambda(\xi, \tau) + \frac{1}{\lambda} f(m^\lambda(\xi, \tau)) \right) . \tag{1.2}$$

Following Pego [13], we will be studying solutions of the equation (1.2) in the limit as  $\lambda$  tends to zero. If we think of  $\lambda$  as representing the inverse of a large length scale, the variable  $\xi$  will be dimensionless, and

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\* The choice of the nonlinearity  $\frac{1}{4}(m^2 - 1)^2$  is used only to carry out the explicit computations in Section 3 and Section 4. Different choices can be made, provided they have the form of a double well potential with equal absolute minima and are smooth enough.

indeed, one often refers to the components of  $\xi$  as being “dimensionless variables”. The dimensionless variables are “slow” and the original variables “fast” for small  $\lambda$ . In what follows we keep the notation  $\xi$  for the slow spatial variables, but drop the use of  $\tau$  and replace it by  $t$  for convenience. One should just bear in mind that now we are looking at the evolution over a *very* long time scale when  $\lambda$  is small. For the reasons indicated above, we shall consider initial data  $m_0(\xi)$  that is  $-1$  in the region bounded by a smooth closed curve  $\Gamma_0$  in  $\Omega$ , and  $+1$  outside this region. We refer to such initial data as “sharp interface initial data”. At later times  $t$  there will still be a fairly sharp interface between a region where  $m(\xi, t) \approx +1$  and where  $m(\xi, t) \approx -1$ , centered on a smooth curve  $\Gamma_t$ . One might hope that for small values of  $\lambda$ , *all information about the evolution on  $m(\xi, t)$  is contained in the evolution of the interface  $\Gamma_t$* . This is indeed the case. To explain the situation more clearly, let  $\mathcal{M}$  denote the set of all smooth simple closed curves in  $\Omega$ . As we will explain in Section 2,  $\mathcal{M}$  can be viewed as a differentiable manifold. A vector field  $V$  on  $\mathcal{M}$  is a functional associating to each  $\Gamma$  in  $\mathcal{M}$  a function in  $C^\infty(\Gamma)$ . This function gives the normal velocity of a point on  $\Gamma$ , and thus describes a “flow” on  $\mathcal{M}$ . We may formally write

$$\frac{d\Gamma_t}{dt} = V(\Gamma_t) . \quad (1.3)$$

Now, given a flow on  $\mathcal{M}$ , we can produce from it an evolution in  $C^\infty(\Omega)$  through the following device: Let  $m$  be any function from  $\mathcal{M}$  to  $C^\infty(\Omega)$ . We write  $m(\xi, \Gamma)$  to denote  $m(\Gamma)$  evaluated at  $\xi \in \Omega$ . We can then define a time dependent function on  $\Omega$ ,  $m(\xi, t)$ , through

$$m(\xi, t) = m(\xi, \Gamma_t) . \quad (1.4)$$

Notice that time dependence in  $m(\xi, t)$  enters *only* through the evolution of  $\Gamma_t$ . A simple example of such a function is the following: Let  $d(\xi, \Gamma)$  denote the signed distance from  $\xi$  to  $\Gamma$ , where the sign is negative in case  $\xi$  is in the interior of  $\Gamma$ , and positive in case  $\xi$  is in the exterior of  $\Gamma$ . The signed distance function, unlike the distance function itself, is smooth near  $\Gamma$ . Let  $g$  be any smooth function on  $\mathbb{R}$  and define

$$m(\xi, \Gamma) = g(d(\xi, \Gamma)) .$$

All the functions appearing in this paper are essentially of this type, or only slightly more elaborate.

Now if, for small  $\lambda$  and sharp interface initial data, all of the information about the evolution of solutions of the Cahn–Hilliard equation were contained in the motion of the interface, then one might hope to find a vector field  $V$  on  $\mathcal{M}$  governing the evolution of the interface, and a function  $m$  from  $\mathcal{M}$  to  $C^\infty(\Omega)$  so that (1.4) defines the corresponding solution of the Cahn–Hilliard equation.

In this paper, we prove a result of this type. We construct a sequence of vector fields  $V_0, V_1, V_2, \dots$  on  $\mathcal{M}$ , and we construct a sequence of functions  $m_0, m_1, m_2, \dots$  from  $\mathcal{M}$  to  $C^\infty(\Omega)$  out of which one may construct arbitrarily accurate approximate solutions of the Cahn–Hilliard equation. The following is the basic prescription, or *ansatz*, for our construction relating an asymptotic expansion for a flow of curves in  $\Omega$  to an asymptotic expansion for an evolution of functions on  $\Omega$ . (We now give only the first part of the *ansatz*, which is all we need in the next few sections. A second part is specified at the beginning of Section 5.)

**Ansatz – part one:** *Let  $V_0, V_1, V_2, \dots$  be a sequence of vector fields on  $\mathcal{M}$  and  $m_0, m_1, m_2, \dots$  be functions from  $\mathcal{M}$  to  $C^\infty(\Omega)$ . For any given initial interface  $\Gamma_0$  in  $\mathcal{M}$ , and all  $N > 0$ , let  $\Gamma_t^{(N)}$  be the solution of*

$$\frac{d\Gamma_t^{(N)}}{dt} = \left[ \sum_{j=0}^{N-1} \lambda^j V_j \right] \left( \Gamma_t^{(N)} \right) \quad \text{with} \quad \Gamma_0^{(N)} = \Gamma_0 . \quad (1.5)$$

*Then define the function  $m^{(N)}(\xi, t)$  by*

$$m^{(N)}(\xi, t) = m_0\left(\frac{d(\xi, \Gamma_t^{(N)})}{\lambda}\right) + \sum_{j=1}^N \lambda^j m_j(\xi, \Gamma_t^{(N)}) .$$

**Theorem 1.1** *For any  $N > 1$  there are vector fields  $V_j$ ,  $j = 0, \dots, (N-1)$  and functions  $m_j$ ,  $j = 0, \dots, N$  as prescribed in the ansatz having the following properties: Let  $T$  denote the lifetime of the solution of (1.5) in  $\mathcal{M}$ . Then there is a constant  $C_N$  so that for all  $t < T$ ,*

$$\frac{\partial}{\partial t} m^{(N)}(\xi, t) = \Delta \left( -\lambda \Delta m^{(N)}(\xi, t) + \frac{1}{\lambda} f(m^{(N)}(\xi, t)) \right) + \Delta R^{(N)}(\xi, t) \quad (1.6)$$

where

$$\sup_{\xi \in \Omega, t \in [0, T]} |R^{(N)}(\xi, t)| \leq C_N \lambda^{N-1}. \quad (1.7)$$

Finally the sequences of vector fields and functions are essentially uniquely determined: Given  $V_j$  for  $j < k$ , then  $V_k$  is determined up to  $\mathcal{O}(\lambda^{k+1})$ , and similarly given  $m_j$  for  $j < k$ , then  $m_k$  is determined up to  $\mathcal{O}(\lambda^{k+1})$ .

Here and in the following  $\mathcal{O}(\lambda^m)$  denotes terms which are of order  $\lambda^m$  uniformly in all of their variables.

The qualified nature of the uniqueness in the theorem is an indication that there will be choices to be made at every stage of the approximation, and that the manageability of the approximation will depend on how those choices are made. The full result, which amplifies Theorem 1.1, will be given in Theorem 5.3.

The connection between flows of curves and the Cahn–Hilliard equation was first made by Pego [13] using a formal analysis with matched asymptotic expansions. He discovered that in leading order, the evolution of the interface should be governed by the Mullins–Sekerka flow. Naturally enough, the corresponding vector field will be the first term,  $V_0$ , in our expansion in (1.5). The Mullins–Sekerka vector field is defined as follows:

Fix a number  $S > 0$  that will later be interpreted as a “surface tension” and denote by  $K(\xi) \equiv K(\xi, \Gamma)$  the curvature at  $\xi \in \Gamma$ . Then for each  $\Gamma$  in  $\mathcal{M}$ , let  $\mu_{0,0}$  be the solution of

$$\Delta \mu_{0,0}(\xi) = 0 \quad \text{for} \quad \xi \in \Omega \setminus \Gamma \quad (1.8)$$

subject to the boundary conditions

$$\mu_{0,0}(\xi) = S \left( K(\xi) - \frac{2\pi}{|\Gamma|} \right) \quad \text{on} \quad \Gamma \quad \text{and} \quad \frac{\partial}{\partial \nu} \mu_{0,0} = 0 \quad \text{on} \quad \partial \Omega. \quad (1.9)$$

(The role of the double subscript will become clear later in the context of our expansion). Now define  $V_0(\Gamma)$  to be the real valued function on  $\Gamma$  given by

$$V_0(\xi, \Gamma) = \left[ \frac{\partial}{\partial \nu} \mu_{0,0} \right]_{\Gamma}(\xi) \quad \xi \in \Gamma \quad (1.10)$$

where the brackets on the right denote the jump in the normal derivative across  $\Gamma$ . This clearly defines a vector field on  $\mathcal{M}$ , and the flow it generates is known as the Mullins–Sekerka flow. Concerning the existence of the solution of the free boundary problem (1.8), (1.9) and (1.10). Chen [6] established the local (in time) existence of a solution in the two dimensional case and, when the initial curve is nearly circular, the global existence and long time behavior. The local existence of a unique smooth solution in any space dimension has been established in [8]. Introducing an alternative approach, Escher and Simonett [10] established the local existence and uniqueness of classical solution to the Mullins–Sekerka problem in any dimension with and without superficial tension, granted enough regularity for the initial hypersurface.

The linear transformation that transforms  $K(\xi)$  into  $V_0(\xi, \Gamma)$  defines what is known as the Dirichlet–Neumann operator  $\mathcal{T}_\Gamma$ . Using this operator, we may write

$$V_0(\xi, \Gamma) = S\mathcal{T}_\Gamma \left( K(\cdot) - \frac{2\pi}{|\Gamma|} \right) (\xi) \quad \xi \in \Gamma.$$

Relevant and useful properties of  $\mathcal{T}_\Gamma$  are recalled in an appendix; see also [9] for more information.

As it is well known, the Mullins–Sekerka flow conserves the area enclosed by  $\Gamma_t$ , and decreases the arc length of  $\Gamma_t$ . To see this, let  $\Omega_\Gamma^-$  denote the interior of  $\Gamma$ , and let  $\Omega_\Gamma^+$  denote its exterior. It is an easy consequence of Green’s identity that

$$\int_{\Gamma_t} V_0(\eta) dS_\eta = \int_{\Gamma_t} \left[ \frac{\partial}{\partial n} \mu_{0,0} \right]_{\Gamma_t} (\eta) dS_\eta = \int_{\Omega \setminus \Gamma_t} \Delta \mu_{0,0}(\eta) d\eta = 0. \quad (1.11)$$

Therefore, since  $\frac{d}{dt} |\Omega_{\Gamma_t}^+| = \int_{\Gamma_t} V(s, t) ds$ , the Mullins–Sekerka flow conserves the area of  $\Omega_{\Gamma_t}^+$ , and hence  $\Omega_{\Gamma_t}^-$  as well. Furthermore,

$$\frac{d}{dt} |\Gamma_t| = - \int_{\Gamma_t} K(s) V_0(s, t) ds = - \frac{1}{S} \int_{\Gamma_t} \mu_{0,0} \left[ \frac{\partial}{\partial n} \mu_{0,0} \right]_{\Gamma_t} ds = - \frac{1}{S} \int_{\Omega} [\nabla \mu_{0,0}]^2 d\xi$$

so that the Mullins–Sekerka flow diminishes the length of the boundary. Clearly a single sphere or multiple spheres of the same radius are equilibria for this evolution. In this paper, we explicitly compute  $V_1$ , the next correction to  $V_0$ , and show how all higher terms could be computed. The description of  $V_1$ , like that of  $V_0$ , is potential theoretic, and somewhat complicated.

**Theorem 1.2** *The vector field  $V_1$  on  $\mathcal{M}$  giving the next corrections to  $V_0$ , the Mullins–Sekerka vector field, is given by  $V_1 = V_1^{(0)} + \langle V_1 \rangle$ ,*

$$\langle V_1 \rangle_\Gamma = \frac{1}{4|\Gamma|} \int_{\Omega} D_{V_0} \mu_{0,0}(\xi, \Gamma) d\xi, \quad (1.12)$$

$$\begin{aligned} V_1^{(0)}(\xi, \Gamma) &= \frac{1}{4}(2S + C)\mathcal{T}_\Gamma V_0(\xi, \Gamma) - \frac{1}{2}\mathcal{T}_\Gamma \left[ p(\cdot) - \frac{1}{|\Gamma|} \int_{\Gamma} p(\xi) dS_\eta \right] (\xi) \\ &\quad - \langle V_1 \rangle_\Gamma \mathcal{T}_\Gamma \left[ \int_{\Gamma} G(\cdot, \eta) dS_\eta - \frac{1}{|\Gamma|} \int_{\Gamma} \int_{\Gamma} G(\xi, \eta) dS_\eta dS_\xi \right] (\xi). \end{aligned} \quad (1.13)$$

Here,  $\mu_{0,0}(\cdot, \Gamma)$  is the harmonic function in  $\Omega \setminus \Gamma$  defining  $V_0$ , see (1.8) and (1.9),  $S$  and  $C$  are explicit constant computed in Section 3, see (3.29) and (3.47),

$$p(\xi, \Gamma) = \frac{1}{2} \int_{\Omega} G(\xi, \eta) \left[ D_{V_0} \mu_{0,0}(\xi, \Gamma) \right] d\eta, \quad (1.14)$$

and  $D_{V_0} \mu_{0,0}(\xi, \Gamma)$  denotes the rate of change of  $\mu_{0,0}(\xi, \Gamma)$  under the flow induced by  $V_0$ .

A formula for computing  $D_{V_0} \mu_{0,0}(\xi, \Gamma)$ , see (4.12), is derived in Section 4. Though complicated, it reduces the computation to standard potential theoretic integrals over  $\Gamma$ .

Pego’s work relating the Cahn–Hilliard equation and the Mullins–Sekerka flow was made rigorous by Alikakos, Bates and Chen [1]. Their construction also yields high order approximate solutions, but does not yield higher order corrections to the sharp interface flow. Their work, like Pego’s, was based on matched asymptotic expansions. Our approach is modeled on the Hilbert expansion of kinetic theory

[4],[5]. Another alternative to matched asymptotic expansions, for the spherically symmetric case, has been developed by Stoth [15].

The plan of the paper is as follows: In Section 2 we describe  $\mathcal{M}$  and vector fields on  $\mathcal{M}$  in a more precise fashion. In Section 3 we explain the Hilbert expansion, and carry out the computations for the first two terms in explicit detail, proving Theorem 1.2. This gives us, in Section 4, the formula for  $V_1$  mentioned above. Finally, we show that the computations can be carried out to any order, and that they yield approximate solutions of the Cahn–Hilliard equation, as claimed in Theorem 1.1. This is accomplished in the remaining sections. We recall the potential theory and some technical lemmas in an Appendix.

The strategy employed here was devised to treat a non–local variant of the Cahn–Hilliard equation that has been rigorously derived from a scaling limit of a spin system with exchange dynamics and local mean field Kac potentials [11]. The sharp interface limit has been investigated by [12] on a formal level as in Pego’s original work. The present approach was developed to facilitate a rigorous treatment, which shall appear in a forthcoming paper.

Finally, E. Orlandi would like to acknowledge discussions with Giorgio Fusco and Nicholas Alikakos. She further thanks the department of Mathematics of Georgia Tech, where part of the work has been completed, for warm hospitality.

## 2. Vector fields and flows on $\mathcal{M}$

### 2.1 A local coordinate system near $\Gamma \in \mathcal{M}$

Let  $\Gamma$  be a smooth closed simple curve in  $\Omega$ . Let  $s \mapsto \eta(s)$  be any arc length parametrization of  $\Gamma$ . (The position of  $\eta(0)$  on  $\Gamma$  does not matter). In the following, we will often denote by  $s$  the corresponding point on the curve. This slight abuse of notation will prove very convenient. Let  $K(s)$  be the curvature of  $\Gamma$  at  $s$ , and let  $\kappa(\Gamma)$  be given by

$$\kappa(\Gamma) = \max_{s \in \Gamma} |K(s)| . \tag{2.1}$$

Recall that the *signed distance* from  $\xi$  to  $\Gamma$ ,  $d(\xi, \Gamma)$ , is defined so that  $d < 0$  inside  $\Gamma$  and  $d > 0$  outside  $\Gamma$ . As long as  $|d(\xi, \Gamma)| < 1/\kappa(\Gamma)$ , there is a uniquely determined point  $\eta \in \Gamma$  such that

$$|\xi - \eta| = |d(\xi, \Gamma)| ;$$

this is the point in  $\Gamma$  that is closest to  $\xi$ . Therefore, define for all  $0 < \lambda_0 < \frac{1}{\kappa(\Gamma)}$ ,

$$\mathcal{N}(\lambda_0) \equiv \mathcal{N}(\lambda_0, \Gamma) = \{\xi \in \mathbb{R}^d : |d(\xi, \Gamma)| < \lambda_0\} .$$

There is a natural set of coordinates in  $\mathcal{N}(\Gamma)$ : Define  $s(\xi)$  to be the arc length coordinate of the unique point  $\eta$  on  $\Gamma$  that is closest to  $\xi$ . This definition extends the domain of definition of the arc length coordinate from  $\Gamma$  itself to all of  $\mathcal{N}(\lambda_0, \Gamma)$ . For the second coordinate, define

$$z(\xi) = \frac{d(\xi, \Gamma)}{\lambda} .$$

The coordinate transformation  $\xi \mapsto (s, z)$  has a simple inverse:

$$\xi = s(\xi) + z\lambda n(s(\xi)) ,$$

where  $n(s)$  denote the unit outward normal to  $\Gamma$  at  $\eta(s)$ . Notice that a small variation in  $\xi$  produces a small variation in  $s$ , but can produce a large variation in  $z$ . For this reason, we speak of  $s$  as the *slow variable*, and  $z$  as the *fast variable*.

### 2.2 Motion in $\mathcal{M}$

The coordinates that we have just introduced in  $\mathcal{N}(\lambda_0, \Gamma)$  provide the means to give  $\mathcal{M}$  the structure of a differentiable manifold, and to study motions in this manifold. Fix any  $\Gamma \in \mathcal{M}$ , and let  $s \mapsto \eta(s)$  denote the arc length parameterization of  $\Gamma$ , as above. Let  $U_\Gamma$  be the subset of  $\mathcal{M}$  consisting of curves  $\tilde{\Gamma}$  such that for each  $\xi \in \tilde{\Gamma}$ ,  $|d(\xi, \Gamma)| < \kappa(\Gamma)$ . Each  $\tilde{\Gamma} \in U_\Gamma$  has a parametrization

$$s \mapsto \eta(s) + r_{\tilde{\Gamma}}(s)n(s) \quad (2.2)$$

for a uniquely determined smooth function  $r_{\tilde{\Gamma}}$ . The map  $\tilde{\Gamma} \mapsto r_{\tilde{\Gamma}}$  maps  $U_\Gamma$  onto an open subset in  $C^\infty(\Gamma)$ , which can of course be identified with  $C^\infty(S^1)$ , where  $S^1$  is the unit circle. Clearly this map is invertible, and we may regard it as a local coordinate map. For each  $\Gamma$  in  $\mathcal{M}$ , and each  $\epsilon$  with  $0 < \epsilon < \kappa(\Gamma)$ , let  $U_{\Gamma, \epsilon}$  consist of all  $\tilde{\Gamma}$  in  $\mathcal{M}$  so that for each  $\xi$  in  $\tilde{\Gamma}$ ,

$$|d(\xi, \Gamma)| < \epsilon .$$

We take these sets as a basis for the topology on  $\mathcal{M}$ . The local coordinates just introduced are very useful for studying the motion of curves in  $\mathcal{M}$ . Let  $t \mapsto \Gamma_t$  be a continuous map from some open interval about  $t = 0$  into  $\mathcal{M}$  such that  $\Gamma_0 = \Gamma$ . It follows from the continuity that for some  $a > 0$ , and each  $t$  with  $|t| < a$ ,  $\Gamma_t$  has a parameterization

$$s \mapsto \eta(s) + r(s, t)n(s) .$$

In this case, knowledge of the function  $r(s, t)$  and its evolution provides complete knowledge about  $\Gamma_t$  and its evolution. In particular, the function\*

$$V(s) = \left. \frac{\partial}{\partial t} r(s, t) \right|_{t=0} \quad (2.3)$$

can be viewed as the tangent vector to the curve  $t \mapsto \Gamma_t$  in  $\mathcal{M}$  at  $t = 0$ . Hence we view  $V$  the *velocity field* of  $t \mapsto \Gamma_t$  at  $t = 0$ . In this sense we write

$$V = \left. \frac{\partial}{\partial t} \Gamma_t \right|_{t=t_0} , \quad (2.4)$$

and identify the tangent space to  $\mathcal{M}$  at  $\Gamma$  as the set of all smooth real valued functions  $V(s)$  on  $\Gamma$ . Thus, a vector field on  $\mathcal{M}$  is a map  $V$  assigning to each  $\Gamma$  in  $\mathcal{M}$  a smooth real valued function  $s \mapsto V(s, \Gamma)$  on  $\Gamma$ . A sufficiently nice vector field on  $\mathcal{M}$  defines a flow on  $\mathcal{M}$ . Given a vector field  $V$  on  $\mathcal{M}$ , and a path  $t \mapsto \Gamma_t$  in  $\mathcal{M}$ , we say that

$$\frac{\partial}{\partial t} \Gamma_t = V(\Gamma_t) \quad (2.5)$$

in case computing the left hand side in the sense of (2.3) and (2.4) gives the same result as evaluating  $V(\Gamma_t)$  according to whatever rule defines it. Then  $t \mapsto \Gamma_t$  is an integral curve of the flow given by  $\Gamma \mapsto V(\cdot, \Gamma)$ . Further, we denote the lifetime  $T$  of the flow (2.5), starting at  $\Gamma \in \mathcal{M}$  as

$$T = \inf\{t > 0 : \kappa(\Gamma_t) \leq \kappa_0\} \quad (2.6)$$

where  $\kappa_0$  is any arbitrarily chosen positive number so that  $\kappa(\Gamma) \leq \kappa_0 < \infty$ . If  $V(s, \Gamma) = K(s, \Gamma)$ , the curvature at  $s \in \Gamma$ , one obtains the curve shortening *flow by curvature*. The nature of this flow in  $d = 2$  has been completely clarified by Grayson. A more pertinent example of a vector field on  $\mathcal{M}$  is the Mullins–Sekerka vector field that we have described in the previous section.

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\* This function will be differentiable by the implicit function theorem.

There is an obvious but useful decomposition of vector fields on  $\mathcal{M}$ . For any vector field  $V$  on  $\mathcal{M}$ , define

$$\langle V \rangle_\Gamma = \frac{1}{|\Gamma|} \int_\Gamma V(s, \Gamma) ds \quad (2.7)$$

and

$$V^{(0)}(s, \Gamma) = V(s, \Gamma) - \langle V \rangle_\Gamma .$$

This gives us the decomposition

$$V(\cdot, \Gamma) = V^{(0)}(\cdot, \Gamma) + \langle V \rangle_\Gamma . \quad (2.8)$$

In this decomposition,  $\langle V \rangle_\Gamma$  is constant, while  $V^{(0)}$  is orthogonal to the constants, and thus generates a flow that does not alter the enclosed area. In what follows, we shall derive separate equations for the components  $V^{(0)}(\cdot, \Gamma)$  and  $\langle V \rangle_\Gamma$  for each of the vector fields  $V_j$  in the ansatz.

We close the section giving another example of a class functions from  $\mathcal{M}$  to  $C^\infty(\Omega)$ :

Let some number  $\lambda_0 > 0$  be given. Define

$$h(\cdot, \Gamma) = \begin{cases} h\left(\frac{d(\xi, \Gamma)}{\lambda}, s(\xi, \Gamma)\right) & \text{when } d(\xi, \Gamma) \leq \lambda_0 \\ 0 & \text{otherwise} \end{cases} \quad (2.9)$$

where  $h$  is a  $C^\infty$  function on  $\mathbb{R} \times \Gamma$ . The functions from  $\mathcal{M}$  to  $C^\infty(\Omega)$  that we use in the ansatz are all functions of this type, or else potentials of them.

### 3. The Hilbert Expansion

#### 3.1 The Hilbert expansion to first order and the Mullins–Sekerka flow

Pego's heuristic demonstration of the relation between the Cahn–Hilliard equation and the Mullins–Sekerka flow, as well as subsequent rigorous work, was carried out in the framework of matched asymptotic expansions. Our approach is based on a Hilbert expansion, adapted from kinetic theory. Nonetheless, the first step is the same: The first step, following Pego, is to write the Cahn–Hilliard equation as a system: For each  $\xi \in \Omega$  and each  $t > 0$ ,

$$\frac{\partial}{\partial t} m^\lambda(\xi, t) = \Delta \mu^\lambda(\xi, t) \quad (3.1)$$

$$\mu^\lambda(\xi, t) = -\lambda \Delta m^\lambda(\xi, t) + \frac{1}{\lambda} f(m^\lambda(\xi, t)) \quad (3.2)$$

where  $\Delta$  denotes the Neumann Laplacian on  $\Omega$ . Let  $\Gamma_0$  be a smooth closed simple curve in  $\Omega$ , and consider initial data  $m^\lambda(\xi, 0)$  such that  $m^\lambda(\xi, 0) \simeq -1$  in the region enclosed by  $\Gamma_0$  and  $m^\lambda(\xi, 0) \simeq +1$ , outside  $\Gamma_0$ . The precise profile of  $m^\lambda(\xi, 0)$  across  $\Gamma_0$  will be specified later. Because the free energy decreases under the evolution described by the Cahn–Hilliard equation, we expect that for initial data that is very close to  $+1$  outside  $\Gamma_0$  and to  $-1$  inside, the solution  $m(\xi, t)$  will remain very close to  $+1$  outside some new curve  $\Gamma_t$  and to  $-1$  inside. We seek an approximate solution  $m_1$  of the form (to be explained below)

$$m_1(\xi, t) = m_0\left(\frac{d(\xi, \Gamma_t)}{\lambda}\right) + \lambda h_1\left(\frac{d(\xi, \Gamma_t)}{\lambda}, s(\xi, \Gamma_t)\right) + \lambda \phi_1(\xi, \Gamma_t) , \quad (3.3)$$

together with an approximate chemical potential  $\mu_0$  so that (3.1) and (3.2) are satisfied to leading order in  $\lambda$ :

$$\frac{\partial}{\partial t} m_1(\xi, t) = \Delta \mu_0(\xi, t) + \mathcal{O}(\lambda) \quad (3.4)$$

and

$$\mu_0(\xi, t) = -\lambda \Delta m_1(\xi, t) + \frac{1}{\lambda} f(m_1(\xi, t)) + \mathcal{O}(\lambda). \quad (3.5)$$

We have in mind expansions for  $m^\lambda$  and  $\mu^\lambda$  of the form

$$m^\lambda = m_0 \left( \frac{d(\cdot, \Gamma_t)}{\lambda} \right) + \sum_{k \geq 1} \lambda^k (h_k + \phi_k) \quad \text{and} \quad \mu^\lambda = \sum_{k \geq 0} \lambda^k \mu_k,$$

of which  $m_1$  and  $\mu_0$  are simply the leading order. In the following sections, we consider expansions of higher order, but it is (3.4) and (3.5) that force the motion of the interface  $\Gamma_t$  to be given by the Mullins–Sekerka flow, at least in leading order. The particular prescription for the form of  $m_1$  requires some explanation. The first term on the right side of (3.3) is the easiest to explain. The function  $m_0$  will be a minor modification of the free energy minimizing transition profile across a planar interface. The minimizing transition profile, which we denote by  $\bar{m}$ , is the unique solution of

$$\begin{aligned} -m''(z) + f(m(z)) &= 0 \quad \text{for} \quad z \in \mathbb{R} \\ \lim_{z \rightarrow \pm\infty} m(z) &= \pm 1 \quad m(0) = 0. \end{aligned} \quad (3.6)$$

It is easy to see that for  $f(m) = m^3 - m$ ,  $\bar{m}(z) = \tanh(z/\sqrt{2})$  and so

$$\bar{m}'(z) > 0, \quad |\bar{m}(z) \pm 1| \leq C_0 e^{-\alpha|z|}, \quad \left| \frac{d^\ell}{dz^\ell} \bar{m}(z) \right| \leq C_\ell e^{-\alpha|z|}, \quad \ell = 1, 2, \dots \quad (3.7)$$

for all  $z \in \mathbb{R}$ , where  $\alpha = \sqrt{2}$  and  $C_\ell$ ,  $\ell = 0, 1, \dots$  are positive real constants. For more general double well potentials, the same sorts of bounds would hold with different  $\alpha$ ; see [14]. It is these sorts of bounds that we will use, and not really the explicit formula for  $\bar{m}(z)$ .

A natural first approximation to a solution of (1.2) with an interface at  $\Gamma_t$  would be

$$\bar{m} \left( \frac{d(\xi, \Gamma_t)}{\lambda} \right).$$

However, this would not define a  $C^\infty$  function. We can remedy this as follows. Fix a number  $\lambda_0$ . Let  $r(u)$  be a smooth, even, unimodal cut-off function so that  $r(u) = 1$  for  $|u| < 1/2$ , and  $r(u) = 0$  for  $u > 1$ . For  $\lambda < \lambda_0$ , define

$$m_0(z) = r \left( \frac{\lambda}{\lambda_0} z \right) \bar{m}(z) + \left( 1 - r \left( \frac{\lambda}{\lambda_0} z \right) \right) \text{sgn}(z). \quad (3.8)$$

Notice that for  $|z| < \lambda_0/(2\lambda)$ ,  $m_0(z) = \bar{m}(z)$ , and for other values of  $z$ , the difference is exponentially small in  $\frac{\lambda}{\lambda_0}$  because of the bounds (3.7). As long as  $\kappa(\Gamma) < 1/\lambda_0$ , perpendicular lines through  $\Gamma$  meet only at points that are at a distance from  $\Gamma$  that is greater than  $\lambda_0$ , and no singularities arise. In what follows, whenever  $m_0$  is used to denote a function on  $\mathbb{R}$ , it will be this function defined in (3.8). As a function from  $\mathcal{M}$  to  $C^\infty(\Omega)$ , it will always denote  $m_0 \left( \frac{d(\xi, \Gamma_t)}{\lambda} \right)$ , the first term on the right in (3.3).

The next two terms on the right side of (3.3) require more explanation. The functions  $h_1$  and  $\phi_1$  give important corrections to the leading term  $m_0 \left( \frac{d(\xi, \Gamma_t)}{\lambda} \right)$ . Long range corrections are given by  $\phi_1$ . It will be a smooth function with derivatives of all orders, and will satisfy a Lipschitz bound that is *independent of*  $\lambda$ . Close to the surface, more rapidly varying corrections may be required, and if

so, these are to be encoded in  $h_1$ . We shall derive an equation for  $h_1$ , and as a consequence of this equation, shall see that  $h_1$  is a rapidly decaying function of  $z$ , like  $m'_0$  above.

As we shall see, there is essentially only one way to choose the motion of  $\Gamma_t$ ,  $m_0$ ,  $h_1$  and  $\phi_1$  so that a solution of (3.4) and (3.5) is possible. As discovered by Pego [13], the motion of  $\Gamma_t$  must be, to leading order, given by the Mullins–Sekerka flow. We will extend this approximation scheme to second order, at which level we compute a first order correction to the Mullins–Sekerka flow, and then from Section 5 on, we show that it may be extended to arbitrary order.

First, however, we explain how (3.4) and (3.5) lead to the Mullins–Sekerka flow. We begin by using (3.1) and (3.3) to determine a leading order approximation to  $\mu$ . To leading order,

$$\frac{\partial}{\partial t} m(\xi, t) \approx \frac{1}{\lambda} m'_0 \left( \frac{d(\xi, \Gamma_t)}{\lambda} \right) V_0(s(\xi)) , \quad (3.9)$$

and we wish to extract the leading order of  $\mu^\lambda$  from (3.1) and this approximation. First recall that the Cahn–Hilliard equation is conservative in that  $\int_\Omega m(\xi, t) d\xi$ , does not depend on  $t$ . If this conservation law is to hold at every order, we would require

$$\int_\Omega \left( \frac{1}{\lambda} m'_0 \left( \frac{d(\xi, \Gamma_t)}{\lambda} \right) V_0(s(\xi)) \right) d\xi = 0 . \quad (3.10)$$

Since

$$d^2\xi = \lambda(1 - \lambda z K(s)) ds dz , \quad (3.11)$$

and since  $m'_0$  is even, this holds if and only if

$$\int_{\Gamma_t} V_0(s, t) ds = 0 \quad \text{for all } t \geq 0 . \quad (3.12)$$

This of course corresponds to the fact that the flow will not change the area enclosed by  $\Gamma_t$ . We therefore suppose that  $V_0$  satisfies (3.12), and we can now find the chemical potential  $\mu$  to leading order: The condition (3.10) is the solvability condition of

$$\frac{1}{\lambda} m'_0 \left( \frac{d(\xi, \Gamma_t)}{\lambda} \right) V_0(s(\xi), t) = \Delta \mu_0(\xi, t) , \quad (3.13)$$

the equation for the leading term in the chemical potential  $\mu^\lambda$  that we have obtained from (3.1) using (3.9). Therefore, by (3.13), we have that to leading order  $\mu^\lambda$  is given by

$$\mu_0(\xi, t) = \int_\Omega G(\xi, \eta) \left( \frac{1}{\lambda} m'_0 \left( \frac{d(\eta, \Gamma_t)}{\lambda} \right) V_0(s(\eta), t) \right) d\eta + c_0(t) \quad (3.14)$$

where  $G(\xi, \eta)$  is the Neumann Greens function for  $\Omega$  (see the appendix), and  $c_0(t)$  is a constant (in  $\xi$ ) to be determined.

Since  $\frac{1}{\lambda} m'_0 \left( \frac{x}{\lambda} \right) \approx 2\delta(x)$ , (3.13) says that  $\mu_0$  itself is approximately equal to a single layer potential plus a time dependent constant  $c_0(t)$ :

$$\mu_{0,0}(\xi, \Gamma_t) = 2 \int_{\Gamma_t} G(\xi, \eta) V_0(\eta, \Gamma_t) dS_\eta + c_0(t) . \quad (3.15)$$

Because  $\mu_0$  is a “smeared” version of  $\mu_{0,0}$ , it will be  $C^\infty$ , unlike  $\mu_{0,0}$  which will only be Lipschitz, with a jump in the normal derivative across  $\Gamma_t$ . However, the only quantitative smoothness bound we have

on  $\mu_0$  that is independent of  $\lambda$  is that it is globally Lipschitz: There is a constant  $C$  depending only on  $\Gamma_t$  so that

$$\|\mu_0\|_{\text{Lip}(\Omega)} \leq C \quad (3.16)$$

We can now explain why we have written the first order corrections to  $m_0$  as a sum of two terms,  $\lambda h_1 + \lambda \phi_1$ . The point is that  $\mu_0$ , being an approximate single layer potential, cannot decay rapidly to a constant: Single layer potentials decay quite slowly, especially in  $\mathbb{R}^2$ . Therefore, we split the correction into two pieces: One, given by  $\phi_1$  will be long range, and will have a purely potential theoretic origin and specification. It will however be defined in the whole domain, not just in an “outer layer”. The function  $h_1$  will provide corrections to  $m_0 + \lambda \phi_1$  that are required near  $\Gamma$ . As we shall see, such corrections are *only* required very near  $\Gamma$ ; the equation determining  $h_1$  shall force its support to be exponentially localized near  $\Gamma$ . We now use (3.5) to determine  $\phi_1$ . Assuming that  $\phi_1$  encodes all first order long range corrections to  $m_0$ , so that  $m'_0$  and  $h_1$  decay rapidly for  $\xi$  far away from  $\Gamma_t$ ,

$$m_1(\xi, t) \approx \pm 1 + \lambda \phi_1(\xi, t) \quad \text{and} \quad \Delta m_1(\xi, t) \approx \lambda \Delta \phi_1(\xi, t) . \quad (3.17)$$

Now consider (3.5). Because of (3.17), and because of the rapid decay of  $h_1$ , for  $\xi$  far from  $\Gamma_t$ ,

$$\mu(\xi, t) \approx \frac{1}{\lambda} f(\pm 1 + \lambda \phi_1(\xi, t)) .$$

To leading order in  $\lambda$ ,  $f(\pm 1 + \lambda \phi_1(\xi, t)) = \lambda f'(1) \phi_1(\xi, t)$ . Hence we must have

$$\phi_1(\xi, t) = \frac{1}{f'(1)} \mu_0(\xi, t) \quad (3.18)$$

for  $\xi$  such that  $|d(\xi, \Gamma_t)| > 1/\kappa(\Gamma_0)$ . This specifies  $\phi_1$  away from  $\Gamma_t$ . It will prove very convenient to take this as the *global* definition of  $\phi_1$ , which we do. Closer to  $\Gamma_t$ , it is the job of  $h_1$  to provide further short range corrections – should these turn out to be needed. Hence we take  $\phi_1$  to be defined globally in  $\Omega$  by (3.18). It then follows immediately from (3.16) that

$$\|\phi_1\|_{\text{Lip}(\Omega)} \leq \frac{C}{f'(1)} . \quad (3.19)$$

Moreover, with this definition

$$\lambda \Delta(\lambda \phi_1(\xi, t)) = \frac{\lambda}{f'(1)} m'_0 \left( \frac{d(\xi, \Gamma_t)}{\lambda} \right) V_0(s(\xi), \Gamma_t) . \quad (3.20)$$

Now we examine (3.2) close to  $\Gamma_t$  to deduce an equation for  $h_1$  and the motion of  $\Gamma_t$ . Since in (3.2) time enters simply as a parameter we avoid writing it in the following, when no confusion arises. We need to express the Laplacian in the  $(z, s)$  coordinate system. This is easily worked out to be

$$\lambda^2 \Delta f = (f_{zz} + \lambda^2 f_{ss}) - \lambda K(s) f_z - \lambda^2 K^2(s) z f_z + \mathcal{O}(\lambda^3) . \quad (3.21)$$

Using (3.21), we easily compute  $\lambda \Delta m_1$  to  $\mathcal{O}(\lambda)$ . Note that because of (3.20), the term in (3.2) involving  $\Delta \phi$  makes no contribution at order  $\lambda^0$ . As for  $m_0$  and  $h_1$ , we have from (3.21) that

$$\lambda^2 \Delta h_1 \left( \frac{d(\xi, \Gamma)}{\lambda}, s(\xi, \Gamma) \right) = \frac{\partial^2}{\partial z^2} h_1(z, s(\xi)) \Big|_{z=d(\xi, \Gamma)/\lambda} + \mathcal{O}(\lambda) .$$

and likewise

$$\lambda \Delta m_0 \left( \frac{d(\xi, \Gamma)}{\lambda} \right) = \left[ \frac{1}{\lambda} \frac{\partial^2}{\partial z^2} - K(s(\xi)) \frac{\partial}{\partial z} \right] m_0(z) \Big|_{z=d(\xi, \Gamma)/\lambda} + \mathcal{O}(\lambda) ,$$

In what follows, we shall use primes to denote derivatives with respect to  $z$ . We obtain from the calculations above and (3.2) that

$$\mu_0(\lambda z, s) = \frac{1}{\lambda} [-m_0''(z) + f(m_0(z))] + [h_1''(z, s) + K(s)m_0'(z) + f'(m_0)(\phi_1 + h_1)] + \mathcal{O}(\lambda) \quad (3.22)$$

First, the term proportional to  $\lambda^{-1}$  in (3.22) must vanish, and so  $m_0$  must satisfy

$$-m_0''(z) + f(m_0(z)) = 0 .$$

This equation is satisfied by the free energy minimizing profile  $\bar{m}$ , and this forces  $m_0$  to be equal to  $\bar{m}$  – up to corrections that are exponentially small in  $\lambda$ . This is the case with  $\bar{m}_0$  as we have defined it.

Introducing the operator  $\mathcal{L}$  defined by

$$\mathcal{L}g(z) = -g''(z) + f'(\bar{m}(z))g(z) , \quad (3.23)$$

we write (3.22), replacing  $m_0$  with  $\bar{m}$  where convenient, as

$$\begin{aligned} \mathcal{L}h_1(z, s) &= \mu_0(s, z) - K(s)m_0'(z) - f'(m_0)\phi + \mathcal{O}(\lambda) \\ &= \left(1 - \frac{f'(m_0(z))}{f'(1)}\right) \mu_0(s, z) - K(s)m_0'(z) + \mathcal{O}(\lambda) . \end{aligned} \quad (3.24)$$

We are now in a position to determine  $h_1$  and  $\phi_1$ . Notice that  $f'(\bar{m})$  is even. In fact, in our case,  $f'(\bar{m}) = 3(\bar{m})^2 - 1$ . This means that  $\mathcal{L}$  is a parity preserving operator; a fact we shall use later on.

Now, with  $F(m) = \frac{1}{4}(m^2 - 1)^2$ ,  $f = F'$  gives

$$f(m) = (m^3 - m), \quad f'(m) = (3m^2 - 1) \quad \text{and} \quad f''(m) = 6m . \quad (3.25)$$

For this potential, we have that  $\bar{m}(x) = \tanh(x/\sqrt{2})$ , and so

$$\bar{m}' = \frac{1}{\sqrt{2}}(1 - \bar{m}^2)$$

Hence it follows that

$$\left(1 - \frac{f'(\bar{m}(z))}{f'(1)}\right) = \frac{3}{2}(1 - \bar{m}^2(z)) = \frac{3}{\sqrt{2}}\bar{m}'(z) .$$

Therefore, (3.24) reduces to

$$\mathcal{L}h_1(z, s) = \left(\frac{3}{\sqrt{2}}\mu_0(s, z) - K(s)\right) \bar{m}'(z) + \mathcal{O}(\lambda) . \quad (3.26)$$

Since  $\bar{m}'(z)$  tends to zero exponentially as  $|z|$  increases, and since  $\mu_0 = \mu_{0,0} + \mathcal{O}(\lambda)$ , and since both are Lipschitz, we finally have

$$\mathcal{L}h_1(z, s) = \left(\frac{3}{\sqrt{2}}\mu_{0,0}(s, 0) - K(s)\right) \bar{m}'(z) + \mathcal{O}(\lambda) . \quad (3.27)$$

The operator  $\mathcal{L}$  is self adjoint on  $L^2(\mathbb{R})$ , and has a null space spanned by  $\bar{m}'$ . Therefore, the condition for solvability of  $\mathcal{L}h_1 = g$  is

$$\int_{\mathbb{R}} g(z)\bar{m}'(z)dz = 0 . \quad (3.28)$$

Evidently this is possible in the case at hand if and only if

$$\mu_{0,0}(s, 0) = SK(s), \quad S = \frac{1}{4} \int_{\mathbb{R}} (\bar{m}'(z))^2 dz = \frac{\sqrt{2}}{3} \quad (3.29)$$

in which case the right hand side of (3.27) vanishes up to  $\mathcal{O}(\lambda)$ , and so we take  $h_1 \equiv 0$ . In other words, the compatibility condition (3.28) forces (3.29) and allows us to take  $h_1 = 0$ , so that there is no short range correction at the first order \*. (Short range corrections will be required at higher orders).

Next, we identify  $V_0$ : It is clear from (3.29) that  $\mu_{0,0}$  is the Dirichlet extension of  $SK$  on  $\Gamma$ , with  $S = \sqrt{2}/3$ . By standard elements of the theory of single layer potentials (see the the appendix) and their Dirichlet data,

$$\mu_{0,0}(\xi, \Gamma) - \frac{1}{|\Gamma|} \int_{\Gamma} \mu_{0,0}(\eta) d\eta = \int_{\Gamma} G(\xi, \eta) V_0(\eta) dS_{\eta} - \frac{1}{|\Gamma|} \int_{\Gamma} \int_{\Gamma} G(\xi, \eta) V_0(\eta) dS_{\eta} dS_{\xi} \quad \xi \in \Omega \quad (3.30)$$

where, with  $\mathcal{T}_{\Gamma}$  denoting the Dirichlet–Neumann operator for  $\Gamma$ ,

$$V_0(\xi) = S\mathcal{T}_{\Gamma} \left( K(\cdot) - \frac{1}{|\Gamma|} \int_{\Gamma} K(s) ds \right) (\xi) . \quad (3.31)$$

Comparing with (3.15), we see that (3.31) determines the velocity field, and we recognize it as the Mullins–Sekerka flow. Next, since for any simple closed curve  $\Gamma$ ,  $\int_{\Gamma} K(s) ds = 2\pi$ , we have

$\int_{\Gamma} \mu_{0,0}(\xi, \Gamma) dS_{\xi} = 2\pi S$ . Therefore, (3.30) becomes

$$\mu_{0,0}(\xi, \Gamma) = \int_{\Gamma} G(\xi, \eta) V_0(\eta) dS_{\eta} - \frac{1}{|\Gamma|} \int_{\Gamma} \int_{\Gamma} G(\xi, \eta) V_0(\eta) dS_{\eta} dS_{\xi} + \frac{2\pi S}{|\Gamma|} \quad \xi \in \Omega .$$

Since we require that  $\mu_0$  simply be a “smeared” version of  $\mu_{0,0}$ , we must use this same constant as the constant  $c_0(t)$  in (3.14). We finally have that

$$\begin{aligned} \mu_0(\xi, \Gamma) &= \int_{\Omega} G(\xi, \eta) \left( \frac{1}{\lambda} m'_0 \left( \frac{d(\eta, \Gamma_t)}{\lambda} \right) V_0(s(\eta), t) \right) d\eta \\ &\quad - \frac{1}{|\Gamma|} \int_{\Gamma} \int_{\Gamma} G(\xi, \eta) V_0(\eta) dS_{\eta} dS_{\xi} + \frac{2\pi S}{|\Gamma|} \quad \xi \in \Omega . \end{aligned} \quad (3.32)$$

and of course

$$\phi_1(\xi, \Gamma) = \frac{1}{f'(1)} \mu_0(\xi, \Gamma) = \frac{1}{2} \mu_0(\xi, \Gamma) . \quad (3.33)$$

Now that  $\phi_1$  is determined, the approximate solution  $m_1(\xi, t)$  in (3.3) is completely specified.

### 3.2 Relation with the Hilbert expansion of kinetic theory

At this point, the analogy with the Hilbert expansion in kinetic [5] theory can be made clear. In this analogy, the Cahn–Hilliard equation corresponds to the Boltzmann equation

$$\frac{\partial f}{\partial t} + \nabla_x \cdot (vf) = \frac{1}{\lambda} \mathcal{Q}(f, f)$$

---

\* This is a consequence of the choice  $f(m) = (m^3 - m)$ .

with a small parameter  $\lambda$ . When  $\lambda$  is small, one must have  $\mathcal{Q}(f, f) \approx 0$  and so  $f \approx M$ , a “local Maxwellian” density on phase space. This has the form

$$M(x, v, t) = \rho(x, t) \left( \frac{1}{2\pi\theta(x, t)} \right)^{3/2} e^{-|v-u(x, t)|^2/2\theta(x, t)}$$

In our problem, the function  $\bar{m} \left( \frac{d(\xi, \Gamma_t)}{\lambda} \right)$  plays the role of a local Maxwellian. In the kinetic theory problem, to determine the evolution of the local Maxwellian, one just needs to determine the evolution of the “hydrodynamic moments”  $\rho(x, t)$ ,  $u(x, t)$  and  $\theta(x, t)$ . The functions “center” the local Maxwellian in exactly the same way that  $\Gamma_t$  centers the front  $m_0 \left( \frac{d(\xi, \Gamma_t)}{\lambda} \right)$  in our problem. In the Hilbert expansion, to leading order, one writes

$$f = M(1 + \lambda h)$$

and seeks a solution of the equation powers of  $\lambda$  just as we did here. The Fredholm criterion provides a compatibility condition for solving an equation involving the linearized Boltzmann operator, and this provides the equations of motion for  $\rho(x, t)$ ,  $u(x, t)$  and  $\theta(x, t)$  just as the compatibility condition for solving an equation involving our operator  $\mathcal{L}$  led to the conclusion that  $\Gamma_t$  evolves under the Mullins–Sekerka flow.

If one continues the Hilbert expansion to higher order, one obtains further refinements to the evolution equations for the hydrodynamical moments: Next come the Navier–Stokes equations, and then the Burnett equations. Continuing it still further, one can construct high order approximate solutions of the Boltzmann equation. These in turn, as was shown by Caflisch [4], can be used to produce solutions of the Boltzmann equation. This has recently been extended to go beyond the appearance of the first shocks by Yu [16]. For recent work on a model with phase segregation, and hence even more directly relevant to the present paper, see [2].

Our goal in the next section is to push this analogy further, and to obtain higher order corrections to the evolution of  $\Gamma_t$ , and higher order approximate solutions of the Cahn–Hilliard equation, sufficient for showing that the unique solution of the Cahn–Hilliard equation with initial data representing phase segregation with a smooth interface has, for later times that are  $\mathcal{O}(1)$ , a smooth interface that has evolved according to the Mullins–Sekerka flow.

We next explicitly carry out the second order expansion, and then prove that the expansion can be continued to arbitrary order.

### 3.3 The prescription at second order

In this section, we prove Theorem 1.2. For this purpose, we seek an approximate solution  $m$  of the Cahn–Hilliard equation of the form

$$m_2(\xi, t) = m_0 \left( \frac{d(\xi, \Gamma_t)}{\lambda} \right) + \lambda \phi_1(\xi, \Gamma_t) + \lambda^2 [h_2(\xi, \Gamma_t) + \phi_2(\xi, \Gamma_t)] \quad (3.34)$$

where  $\phi_1$  is the function determined in the previous section, and  $h_2$  and  $\phi_2$  are to be determined here, so that (3.1) and (3.2) is satisfied to  $\mathcal{O}(\lambda^2)$ . This time, we will require a short range correction, and  $h_2$  will not vanish. It is worth doing the expansion to second order explicitly. One reason is that new features concerning the compatibility conditions enter at second order, but after that, the pattern is essentially the same. The second reason is that this provides the form of the leading corrections to the Mullins–Sekerka flow.

Indeed, to carry out the expansion to second order, we let  $\Gamma_t^{(1)}$  denote the solution to

$$\frac{\partial}{\partial t} \Gamma_t^{(1)} = V_0(\Gamma_t^{(1)}) + \lambda V_1(\Gamma_t^{(1)}) \quad , \quad \Gamma_0^{(1)} = \Gamma_0 \quad , \quad (3.35)$$

where  $V_0$  is the Mullins–Sekerka vector field on  $\mathcal{M}$ , and  $V_1$ , as it will be soon explained, is to be determined by two different types of compatibility conditions. Our first step will be to determine a higher order approximate chemical potential using (3.1). Keeping terms out to first order in  $\lambda$  in both  $m$  and the chemical potential  $\mu$ , we have the equation:

$$\frac{\partial}{\partial t} \left( m_0 \left( \frac{d(\xi, \Gamma_t^{(1)})}{\lambda} \right) + \lambda \phi_1(\xi, \Gamma_t^{(1)}) \right) = \Delta(\mu_0 + \lambda \mu_1) . \quad (3.36)$$

The quantity  $\phi_1(\xi, \Gamma_t^{(1)})$  is not so easy to differentiate, even apart from the fact that  $\Gamma_t^{(1)}$  is evolving under  $V_0 + \lambda V_1$ . We must first obtain an equation specifying  $V_1$ , and for this purpose, we must isolate the leading contribution from the evolution under  $V_0$ : For each  $t$ , let  $\tilde{\Gamma}_{t+s}$  be given by

$$\frac{d}{ds} \tilde{\Gamma}_{t+s} = V_0(\tilde{\Gamma}_{t+s}) \quad \text{with} \quad \tilde{\Gamma}_t = \Gamma_t^{(1)} .$$

We then define  $D_{V_0} \phi_1$  by

$$D_{V_0} \phi_1(\xi, \Gamma_t^{(1)}) = \lim_{s \rightarrow 0} \frac{1}{s} \left( \phi_1(\xi, \tilde{\Gamma}_{t+s}) - \phi_1(\xi, \tilde{\Gamma}_t) \right) . \quad (3.37)$$

This way,

$$\frac{\partial}{\partial t} \phi_1(\xi, \Gamma_t^{(1)}) = D_{V_0} \phi_1(\xi, \Gamma_t^{(1)}) + \mathcal{O}(\lambda)$$

since in computing  $D_{V_0} \phi_1(\xi, \Gamma_t^{(1)})$  we have only suppressed  $\lambda V_1$ . We therefore replace (3.36) by

$$\frac{\partial}{\partial t} \left( m_0 \left( \frac{d(\xi, \Gamma_t^{(1)})}{\lambda} \right) \right) + \lambda D_{V_0} \phi_1(\xi, \Gamma_t^{(1)}) = \Delta(\mu_0 + \lambda \mu_1) . \quad (3.38)$$

The compatibility condition for the solvability of (3.38) is that

$$\frac{d}{dt} \int_{\Omega} \left( m_0 \left( \frac{d(\xi, \Gamma_t^{(1)})}{\lambda} \right) \right) d\xi + \lambda \int_{\Omega} D_{V_0} \phi_1(\xi, \Gamma_t^{(1)}) d\xi = 0 . \quad (3.39)$$

As one sees from the formula (3.32) for  $\phi_1$ , there is no reason that  $\int_{\Omega} D_{V_0} \phi_1(\xi, \Gamma_t^{(1)}) d\xi$  will vanish in general. We shall deduce a formula for this quantity in the next section, but what is relevant now is that it must be cancelled by the term in (3.39) involving  $m_0$ . When  $\Gamma_t^{(1)}$  evolves under (3.35), we have that

$$\frac{\partial}{\partial t} m_0 \left( \frac{d(\xi, \Gamma_t^{(1)})}{\lambda} \right) = \frac{1}{\lambda} m'_0 \left( \frac{d(\xi, \Gamma_t^{(1)})}{\lambda} \right) [V_0(s(\xi)) + \lambda V_1(s(\xi))] .$$

Integrating the right hand side over  $\Omega$ , we find

$$\int m'_0(z) dz \int_{\Gamma_t^{(1)}} [V_0(s) + \lambda V_1(s)] ds = 2\lambda \int_{\Gamma_t^{(1)}} V_1(s) ds .$$

Hence, the compatibility condition for solvability of (3.36) will hold if and only if

$$2 \int_{\Gamma_t^{(1)}} V_1(s) ds = - \int_{\Omega} D_{V_0} \phi_1(\xi, \Gamma_t^{(1)}) d\xi . \quad (3.40)$$

We therefore decompose  $V_1$  into two pieces  $V_1 = V_1^{(0)} + \langle V_1 \rangle_\Gamma$  as in (2.8). The part  $\langle V_1 \rangle_\Gamma$  denotes the average of  $V_1$  over  $\Gamma$ . It's constant value is determined by (3.40):

$$\langle V_1 \rangle_\Gamma = -\frac{1}{2|\Gamma_t^{(1)}|} \int_\Omega D_{V_0} \phi_1(\xi, \Gamma_t^{(1)}) d\xi . \quad (3.41)$$

where the  $\phi_1$  is given in (3.18). (This is somewhat different from the expression for  $\langle V_1 \rangle_\Gamma$  given in Theorem 1.2, but as we shall see, the difference is  $\mathcal{O}(\lambda)$ ).

The part  $V_1^{(0)}$  will be orthogonal to the constants. It will be determined by a compatibility condition that arises when we solve for  $h_2$ . The pattern is the same in all higher orders: The constant part of the  $k$ th order velocity field,  $\langle V_k \rangle_\Gamma$  will be determined by the compatibility condition needed to solve Laplace's equation for  $\mu_k$ . The non-constant part  $V_k^{(0)}$  will be determined by the compatibility condition needed to solve for  $h_k$ . Since  $\langle V_1 \rangle_\Gamma$  is not zero, the area enclosed by  $\Gamma_t^{(1)}$  as it evolves under (3.35) will not be constant. This should not be surprising. Only at the sharp interface limit does the conservation of  $\int_\Omega m(\xi, t) d\xi$  coincide with the conservation of the area enclosed by  $\Gamma_t$ . At higher order, the interactions between the shape of the curve and shape of the interface matter. With this choice of  $\langle V_1 \rangle_\Gamma$ , the compatibility condition (3.39) is satisfied and we can now solve (3.38) for  $\mu_0 + \lambda\mu_1$ . First, we use the full evolution, under (3.35), to differentiate the first term in (3.38). Then we apply the Green's function to each of the pieces. Taking in account (3.13) we obtain that

$$\mu_1(\xi, t) = \frac{1}{\lambda} \int_\Omega G(\xi, \eta) m'_0 \left( \frac{d(\xi, \Gamma_t^{(1)})}{\lambda} \right) V_1(s(\eta)) d\eta + p(\xi, t) + c_1(t) \quad (3.42)$$

where we set

$$p(\xi, t) = \int_\Omega G(\xi, \eta) \left[ D_{V_0} \phi_1(\xi, \Gamma_t^{(1)}) \right] d\eta , \quad (3.43)$$

and  $c_1(t)$  is a constant (in  $\xi$ ) to be determined. (Again, this is somewhat different from the expression for  $p(\xi, t)$  given in Theorem 1.2, but as we shall see, the difference is  $\mathcal{O}(\lambda)$ ). In the next section, we shall derive a more explicit formula, at least in terms of potential theory, for  $p(\xi, t)$ . For the time being, it is convenient to work with this compact form. The first term on the right is an approximate single layer potential and it is still to be fully determined, since we do not know  $V_1^{(0)}$  yet.

Toward this end, we first determine  $\phi_2$ . As before, consider  $\xi$  far from  $\Gamma_t^{(1)}$  where  $m_0^2 - 1$  and  $h_2$  are negligible. We then have

$$\mu_0 + \lambda\mu_1 \approx \frac{1}{\lambda} f(1 \pm \lambda\phi_1 + \lambda^2\phi_2) \approx f'(1) (\phi_1 + \lambda\phi_2) + \frac{f''(1)}{2} \lambda\phi_1^2 .$$

Since  $\mu_0 = f'(1)\phi_1$ , we define  $\phi_2$  by  $\phi_2 = \frac{1}{f'(1)} \left[ \mu_1 - \frac{f''(1)}{2(f'(1))^2} \mu_0^2 \right] = \frac{1}{2} \mu_1 + \frac{3}{8} \mu_0^2$ . As before, we use this definition globally in  $\Omega$ , and  $\phi_2$  is Lipschitz with a norm bounded independently of  $\lambda$ . We now return to (3.2), and replace  $\mu^\lambda$  by  $\mu_0 + \lambda\mu_1$  and  $m^\lambda$  by (3.34). From (3.21), and the potential theoretic definition of  $\phi_1$ ,

$$\begin{aligned} & -\lambda\Delta(m_0 + \lambda\phi_1 + \lambda^2h_2 + \lambda^2\phi_2) = \\ & -\frac{1}{\lambda} [m_0'' + Km'_0 + \lambda \left[ -h_2'' + \frac{1}{f'(1)} m'_0 V_0 + K^2 z m'_0 \right]] + \mathcal{O}(\lambda^2) . \end{aligned}$$

Clearly,

$$\begin{aligned} & \frac{1}{\lambda} f(m_0 + \lambda\phi_1 + \lambda^2h_2 + \lambda^2\phi_2) = \\ & \frac{1}{\lambda} f(m_0) + f'(m_0)\phi_1 + \lambda f'(m_0)[h_2 + \phi_2] + \lambda \frac{f''(m_0)}{2} \phi_1^2 . \end{aligned}$$

Hence from (3.2), since  $\bar{m}$  solves (3.6), and since  $m_0 = \bar{m}$  up to exponentially small corrections,

$$\begin{aligned} \mu_0 + \lambda\mu_1 &= [Km'_0 + f'(m_0)\phi_1] \\ &+ \lambda \left[ \mathcal{L}h_2 + K^2zm'_0 + \frac{1}{f'(1)}m'_0V_0 + f'(m_0)\phi_2 + \frac{1}{2}f''(m_0)\phi_1^2 \right] + \mathcal{O}(\lambda^2). \end{aligned}$$

Therefore, since  $\phi_1 = (1/f'(1))\mu_0 = (1/2)\mu_0$ , we can use the identity  $(1-f'(m_0)/f'(1)) = (3/\sqrt{2})m'_0$ , and have

$$\begin{aligned} \mu_1 &= \frac{1}{\lambda} \left( K - \frac{3}{\sqrt{2}}\mu_0 \right) m'_0 \\ &+ \left[ \mathcal{L}h_2 + K^2zm'_0 + \frac{1}{f'(1)}m'_0V_0 + f'(m_0)\phi_2 + \frac{1}{2}f''(m_0)\phi_1^2 \right] + \mathcal{O}(\lambda). \end{aligned} \quad (3.44)$$

Notice that the first term on the right is bounded uniformly in  $\lambda$  because  $\mu_0$  is Lipschitz,  $\mu_0 = \mu_{0,0} + \mathcal{O}(\lambda)$ , and  $m'_0$  decays rapidly. From (3.44) we obtain

$$\begin{aligned} \mathcal{L}h_2 &= \mu_1 - \frac{1}{\lambda} \left( K - \frac{3}{\sqrt{2}}\mu_0 \right) m'_0 \\ &- \left[ K^2zm'_0 + \frac{1}{f'(1)}m'_0V_0 + f'(m_0)\phi_2 + \frac{1}{2}f''(m_0)\phi_1^2 \right] + \mathcal{O}(\lambda). \end{aligned} \quad (3.45)$$

To solve (3.45), the compatibility condition (3.28) need to be satisfied. This condition will determine  $V_1^{(0)}$  and therefore  $\mu_1$  will be fully determined. Denote

$$g_1(s) = \int_{\mathbb{R}} \left[ K^2zm'_0 + \frac{1}{f'(1)}m'_0V_0 + f'(m_0)\phi_2 + \frac{1}{2}f''(m_0)\phi_1^2 \right] m'_0 dz$$

Since  $m'_0(z)$  is even,  $f''(m_0) = 6m_0$ ,  $\phi_1$  and  $\phi_2$  have a Lipschitz bound independently on  $\lambda$  we obtain

$$\begin{aligned} g_1(s) &= \int_{\mathbb{R}} \left[ \frac{1}{f'(1)}m'_0V_0 + f'(m_0)\phi_2(0, s) + \frac{1}{2}f''(m_0)\phi_1^2(0, s) \right] m'_0 dz + \mathcal{O}(\lambda) \\ &= 2SV_0(s) + \mathcal{O}(\lambda). \end{aligned} \quad (3.46)$$

where  $S$  is the surface tension defined in (3.29). We next investigate

$$g_2(s) = \frac{1}{\lambda} \int_{\mathbb{R}} \left( K - \frac{3}{\sqrt{2}}\mu_0 \right) (m'_0)^2(z) dz$$

Then

$$g_2(s) = \frac{1}{\lambda} \int_{\mathbb{R}} \left( K - \frac{3}{\sqrt{2}}\mu_{0,0} \right) (m'_0)^2(z) dz - \frac{3}{\sqrt{2}\lambda} \int_{\mathbb{R}} (\mu_0 - \mu_{0,0}) (m'_0(z))^2 dz .$$

The second term, involving the difference between the “smeared” and exact single layer potentials is easily seen to be  $\mathcal{O}(\lambda^2)$ . As for the first one, note that we have  $\mu_{0,0}(s, z) = SK + a\lambda z + \mathcal{O}(\lambda^2)$  for  $z > 0$  and  $\mu_{0,0}(s, z) = SK + b\lambda z + \mathcal{O}(\lambda^2)$  for  $z < 0$ . The quantity  $b - a$  is just the jump in the normal derivative of  $\mu_{0,0}$  across the interface at  $s$ , which is  $V_0(s)$ . Hence with  $C$  defined by

$$C = \int_{\mathbb{R}} |z|(m'_0)^2 dz = \frac{4 \ln(2) - 1}{6}, \quad (3.47)$$

we have

$$g_2(s) = CV_0(s) + \mathcal{O}(\lambda) . \quad (3.48)$$

The compatibility condition that we must have in order to solve for  $h_2$  is that

$$\int \mu_1(\lambda z, s)m'_0(z)dz = g_1(s) + g_2(s) , \quad (3.49)$$

where  $\mu_1$  is given in (3.42). It is convenient to single out from  $\mu_1$  the part still unknown which will be determined so that (3.49) holds. We denote

$$\mu_{1,0}(\xi, \Gamma) = \frac{1}{\lambda} \int_{\Omega} G(\xi, \eta)m'_0 \left( \frac{d(\xi, \Gamma)}{\lambda} \right) V_1^{(0)}(s(\eta))d\eta + c_1(t) \quad (3.50)$$

and

$$\tilde{\mu}_1(\xi, \Gamma) = \langle V_1 \rangle \frac{1}{\lambda} \int_{\Omega} G(\xi, \eta)m'_0 \left( \frac{d(\eta, \Gamma)}{\lambda} \right) d\eta + p(\xi, \Gamma) . \quad (3.51)$$

Taking in account (3.46) and (3.48) we then write (3.49) as

$$\int \mu_{1,0}(\lambda z, s)m'_0(z)dz = (2S + C)V_0 - \int \tilde{\mu}_1(\lambda z, s)m'_0(z)dz = (2S + C)V_0 - 2\tilde{\mu}_1(0, s) + \mathcal{O}(\lambda)$$

having  $\tilde{\mu}_1$  a Lipschitz bound independent on  $\lambda$ . As in the previous section, we may substitute  $\mu_{1,0}$  by the corresponding single layer potential

$$\mu_{1,0,0}(\xi, \Gamma) = 2 \int_{\Gamma} G(\xi, \eta)V_1^{(0)}(\eta)dS_{\eta} + c_1(t) \quad (3.52)$$

and further restrict to  $z = 0$ , obtaining

$$\begin{aligned} \int \mu_{1,0,0}(0, s)m'_0(z)dz &= 2\mu_{1,0,0}(0, s) = (2S + C)V_0 - 2\tilde{\mu}_1(0, s) \\ &= (2S + C)V_0 + 4\langle V_1 \rangle_{\Gamma} \int_{\Gamma} G(s, \eta)dS_{\eta} + 2p(s) + \mathcal{O}(\lambda). \end{aligned} \quad (3.53)$$

Inserting (3.52) into (3.53) and integrating over  $\Gamma$  we obtain that

$$\begin{aligned} 2c_1(t) &= -\frac{4}{|\Gamma|} \left\{ \int_{\Gamma} \int_{\Gamma} G(\xi, \eta)V_1^{(0)}(\eta)dS_{\eta}dS_{\xi} + \int_{\Gamma} dS_{\xi} \int_{\Gamma} G(\xi, \eta)dS_{\eta} \right\} \\ &\quad - \frac{2}{|\Gamma|} \int_{\Gamma} p(\eta)dS_{\eta} . \end{aligned} \quad (3.54)$$

In this way  $c_1(t)$  is written in term of  $V_1^{(0)}$ , still to be determined. Taking in account (3.54) we obtain from (3.53) an equation for  $V_1^{(0)}$

$$\begin{aligned} \mathcal{S}_{\Gamma}V_1^{(0)}(\xi) &= \frac{1}{4}(2S + C)V_0(\xi) - \langle V_1 \rangle_{\Gamma} \left[ \int_{\Gamma} G(\xi, \eta)dS_{\eta} - \frac{1}{|\Gamma|} \int_{\Gamma} dS_{\xi} \int_{\Gamma} G(\xi, \eta)dS_{\eta} \right] \\ &\quad - \frac{1}{2} \left[ p(\xi) - \frac{1}{|\Gamma|} \int_{\Gamma} p(\eta)dS_{\eta} \right] , \end{aligned}$$

where  $\mathcal{S}_\Gamma$  is the operator defined in (10.9). Finally applying the Dirichlet-Neumann operator, see (10.8) we obtain

$$\begin{aligned} V_1^{(0)} &= \frac{1}{4}(2S + C)\mathcal{T}_\Gamma V_0 - \langle V_1 \rangle_\Gamma \mathcal{T}_\Gamma \left[ \int_\Gamma G(\cdot, \eta) dS_\eta - \frac{1}{|\Gamma|} \int_\Gamma dS_\xi \int_\Gamma G(\xi, \eta) dS_\eta \right] \\ &\quad - \frac{1}{2} \mathcal{T}_\Gamma \left[ p(\cdot) - \frac{1}{|\Gamma|} \int_\Gamma p(\eta) dS_\eta \right] . \end{aligned} \quad (3.55)$$

With  $V_1 = V_1^{(0)} + \langle V_1 \rangle_\Gamma$  determined, we can solve (3.44) for  $h_2$  and we have our approximation.

At this stage it is a simple matter to prove Theorem 1.2:

**Proof of Theorem 1.2** We have seen that up to an adjustment of size  $\mathcal{O}(\lambda)$ , we must have that  $\langle V_1 \rangle_\Gamma$  is given by (3.41). To see that this agrees, up to an adjustment of size  $\mathcal{O}(\lambda)$ , with the expression (1.12) given in Theorem 1.2, note first that by (3.18), and the fact that  $f'(1) = 2$ ,  $\phi_1 = \mu_0/2$ . It then remains to show that

$$D_{V_0} \mu_0(\xi, \Gamma) = D_{V_0} \mu_{0,0}(\xi, \Gamma) + \mathcal{O}(\lambda) . \quad (3.56)$$

For this purpose, let  $s \mapsto \eta(s)$  be any arclength parameterization of  $\Gamma$ , and let  $\Gamma_z$  denote the curve parametrized by  $s \mapsto \eta(s) + \lambda z n(s)$ , using the notation of Section 2. Then, by the definition of  $\mu_0$ , (3.13), and from (3.11), we have

$$2\mu_0(\xi, \Gamma) = \int_{\mathbb{R}} \mu_{0,0}(\xi, \Gamma_z) m'_0(z) dz + \mathcal{O}(\lambda) ,$$

and

$$2D_{V_0} \mu_0(\xi, \Gamma) = \int_{\mathbb{R}} D_{V_0} \mu_{0,0}(\xi, \Gamma_z) m'_0(z) dz + \mathcal{O}(\lambda) .$$

To draw the desired conclusion, we must know that  $\Gamma \mapsto D_{V_0} \mu_0(\xi, \Gamma)$  is continuous from  $\mathcal{M}$  to, say,  $L^2(\Omega)$ . This can be seen from formula (4.12) in the next section, and hence

$$D_{V_0} \mu_{0,0}(\xi, \Gamma_z) = D_{V_0} \mu_{0,0}(\xi, \Gamma) + \mathcal{O}(\lambda) ,$$

in  $L^2(\Omega)$ , which gives us (3.56).

Next, we have seen that  $V_1^{(0)}$  must be given by (3.55), up to an adjustment of size  $\mathcal{O}(\lambda)$ . This coincides with the formula (1.13) in Theorem 1.2, except that the formulae for  $p$  differ: The formula (1.14) in Theorem 1.2 involves  $D_{V_0} \mu_{0,0}(\xi, \Gamma)$ , while the formula (3.43) for the  $p$  in (3.55) involves  $D_{V_0} \mu_0(\xi, \Gamma)$ . However, by (3.56) once again, these differ by  $\mathcal{O}(\lambda)$ .  $\square$

## 4. Some differentiation formulas

### 4.1 The general problem

In this section, we produce a potential theoretic formula for the rate of change of  $V_0$  under its own time evolution. This permits us to give a potential theoretic formula for the function  $D_{V_0} \phi_1(\xi, \Gamma_t^{(1)})$ , and hence  $p(\xi, t)$ , so that (1.13) becomes more explicit.

From the formula (3.32) for  $\mu_0$  and hence  $\phi_1$ , we see that the main problem to be dealt with here is of the following type: Suppose that we are given two vector fields  $V$  and  $W$  on  $\mathcal{M}$ . Suppose further that the first vector field does not affect the enclosed area; i.e., for all  $\Gamma$ ,  $\int_\Gamma V(s) ds = 0$ . Using  $V$ , form the the Neumann harmonic extension (see the Appendix)

$$\psi_V(\xi, \Gamma) = \int_\Gamma G(\xi, \eta) V(\eta, \Gamma) dS_\eta - \frac{1}{|\Gamma|} \int_\Gamma \int_\Gamma G(\xi, \eta) V(\eta, \Gamma) dS_\eta dS_\xi \quad \xi \in \Omega \quad (4.1)$$

and take

$$\frac{d\Gamma_t}{dt} = W(\Gamma_t), \quad \Gamma_0 = \Gamma. \quad (4.2)$$

We wish to compute  $D_W \psi_V = \frac{d}{dt} \psi_V(\xi, \Gamma_t)$ . In the particular case that  $V$  is  $V_0$ , the Mullins–Sekerka vector field,  $\psi_{V_0}(\xi, \Gamma) = \mu_{0,0}$ , and hence if we take  $W = V_0$  as well, the quantity we are computing is  $D_{V_0} \mu_{0,0}$ , which figures in Theorem 1.2. We first derive a general result. We parameterize  $\Gamma_t$  by the arc length of  $\Gamma$  as follows: Let  $s \mapsto \xi(s)$  be an arc length parametrization of  $\Gamma$ . For  $t$  sufficiently small, every point on  $\Gamma_t$  belongs to  $\mathcal{N}(\lambda_0, \Gamma)$ , for some strictly positive  $\lambda_0 \leq \frac{1}{\kappa(\Gamma)}$ . Hence we can use the coordinates introduced in Section 2 to write a parametrization

$$s \mapsto \xi(s) + r_{\Gamma_t}(s)n(s) \quad 0 \leq s \leq |\Gamma|$$

of  $\Gamma_t$ . Clearly,  $r_{\Gamma_t}(s) = W(s)t + \mathcal{O}(t^2)$ . Let  $V(s, t)$  be the coordinate representation of a vector field  $V(\cdot, \Gamma_t)$  on  $\mathcal{M}$  at  $\Gamma_t$ . That is, if  $n(s, t)$  is the outward normal to  $\Gamma_t$  at  $\xi(s) + r_{\Gamma_t}(s)n(s)$ , then  $V(\cdot, \Gamma_t)$  is given by the vectorfield

$$V(s, t)n(s, t)$$

on  $\Gamma_t$ . We seek a formula for  $\partial V(s, t)/\partial t$ . In the case of the Mullins–Sekerka vector field, and others that we shall encounter here,  $V$  is explicitly defined through the Dirichlet–Neumann operator, or more precisely, its inverse: We are given a function  $f(\cdot, \Gamma)$  such that  $\int_{\Gamma} f(s, \Gamma) ds = 0$ , and then  $V(\cdot, \Gamma) = \mathcal{T}_{\Gamma} f(\cdot, \Gamma)$ . For the Mullins–Sekerka vector field, as we have seen,

$$f(s, \Gamma) = SK(s, \Gamma) - \frac{S}{|\Gamma|} \int_{\Gamma} K(s, \Gamma) ds = S \left( K(s, \Gamma) - \frac{2\pi}{|\Gamma|} \right). \quad (4.3)$$

It is relatively easy to compute the evolution of  $f(\cdot, \Gamma)$  as  $\Gamma$  evolves according to (4.2). We will use this to compute  $\frac{\partial}{\partial t} V(\cdot, t)$  in terms of  $\frac{\partial}{\partial t} f(\cdot, \Gamma_t)$ . In the following we will denote simply by  $K(s) \equiv K(s, \Gamma)$  being  $\Gamma$  the initial curve for the evolution (4.2). The result is the following:

**Theorem 4.1** *Let  $V$  and  $W$  be two smooth vector fields on  $\mathcal{M}$ , and suppose that  $V$  is defined through  $V = \mathcal{T}_{\Gamma_t}(f(\cdot, \Gamma_t))$ . Suppose that  $\Gamma_t$  evolves according to (4.2). Let  $\mathcal{Q}$  be the operator on  $L^2(\Gamma)$  defined by*

$$\mathcal{Q}_{\Gamma, W} h(s) = \int_0^{|\Gamma|} [\nabla_{\xi} G(\xi(s), \eta(r)) \cdot W(s)n(s) + \nabla_{\eta} G(\xi(s), \eta(r)) \cdot W(r)n(r)] h(r) dr. \quad (4.4)$$

Then we have

$$\begin{aligned} & \frac{\partial}{\partial t} V(s, 0) + W(s)K(s)V(s, 0) \\ &= \mathcal{T}_{\Gamma} \left( \frac{\partial}{\partial t} f(s, 0) + \frac{1}{|\Gamma|} \int_0^{|\Gamma|} f(s, 0) W(s) K(s) ds \right) \\ & - \mathcal{T}_{\Gamma} \left( \mathcal{Q}_{\Gamma, W} V(s, 0) - \frac{1}{|\Gamma|} \int_{\Gamma} \mathcal{Q}_{\Gamma, W} V(s, 0) ds \right). \end{aligned} \quad (4.5)$$

Notice that the operator  $\mathcal{Q}_{\Gamma, W}$  is a bounded smoothing operator, the singularities in the two derivatives of the Green’s function cancel.

**Proof:** In the proof, we shall drop some subscripts. A simple computation shows that the element of arc length along  $\Gamma_t$  in the parameterization that we employ is

$$\rho(s, t) ds = \left( (1 + w(s, t)K(s))^2 + (\partial w(s, t)/\partial s)^2 \right)^{1/2} ds \quad 0 \leq s \leq |\Gamma|.$$

Again, it is easy to see that

$$\rho(s, t) = 1 + tW(s)K(s) + \mathcal{O}(t^2) .$$

Since, see (10.10),  $f(\cdot, \Gamma_t) = \mathcal{S}_{\Gamma_t} V(\cdot, \Gamma_t)$  we have

$$\begin{aligned} f(s, t) &= \int_0^{|\Gamma|} G(\xi(s) + w(s, t)n(s), \eta(r) + w(r, t)n(r))V(r, t)\rho(r, t)dr \\ &\quad - \frac{1}{|\Gamma_t|} \int_0^{|\Gamma|} \int_0^{|\Gamma|} G(\xi(s) + w(s, t)n(s), \eta(r) + w(r, t)n(r))V(r, t)\rho(r, t)dr\rho(s, t)ds. \end{aligned}$$

Recalling the definition (4.4) of  $\mathcal{Q}_{\Gamma, W}$ , we have that

$$\begin{aligned} \frac{\partial}{\partial t} f(s, 0) &= \mathcal{S}_{\Gamma} \left( \frac{\partial}{\partial t} V(s, 0) + W(s)K(s)V(s, 0) \right) \\ &\quad + \mathcal{Q}_{\Gamma, W} V(s, 0) - \frac{1}{|\Gamma|} \int_0^{|\Gamma|} \mathcal{Q}_{\Gamma, W} V(s, 0) ds \\ &\quad + \frac{|\Gamma|'}{|\Gamma|^2} \int_0^{|\Gamma|} \int_0^{|\Gamma|} G(\xi(s), \eta(r))V(r, 0) dr ds \\ &\quad - \frac{1}{|\Gamma|} \int_0^{|\Gamma|} W(s)K(s) \left( \int_0^{|\Gamma|} G(\xi(s), \eta(r))V(r, 0) dr \right) ds . \end{aligned}$$

where

$$\left. \frac{d|\Gamma_t|}{dt} \right|_{t=0} \equiv |\Gamma|' = \int_0^{|\Gamma|} K(s)W(s)ds . \quad (4.6)$$

From the definition of  $f(s, t)$  and (4.6) one easily recognizes the contribution of the last two terms as  $-\frac{1}{|\Gamma|} \int_0^{|\Gamma|} f(s, 0)W(s)K(s)ds$ . Then since, by assumption,  $\int_0^{|\Gamma|} f(s, t)\rho(s, t)ds = 0$  identically in  $t$ , we have that

$$\int_0^{|\Gamma|} \frac{\partial}{\partial t} f(s, 0)ds = - \int_0^{|\Gamma|} f(s, 0)W(s)K(s)ds .$$

Combining results, we have the following identity:

$$\begin{aligned} \frac{\partial}{\partial t} f(s, 0) + \frac{1}{|\Gamma|} \int_0^{|\Gamma|} f(s, 0)W(s)K(s)ds \\ = \mathcal{Q}_{\Gamma, W} V(s, 0) - \frac{1}{|\Gamma|} \int_0^{|\Gamma|} \mathcal{Q}_{\Gamma, W} V(s, 0) ds \\ + \mathcal{S}_{\Gamma} \left( \frac{\partial}{\partial t} V(s, 0) + W(s)K(s)V(s, 0) \right) . \end{aligned}$$

Applying the Dirichlet–Neumann operator, see (10.8), we have the result.  $\square$

## 4.2 Application to $D_{V_0} \mu_{0,0}$

Let  $\Gamma$  evolve under (4.2), from (4.3), using (4.6) again, we obtain

$$\frac{\partial}{\partial t} f(s, 0) = S \left. \frac{\partial}{\partial t} K(s, t) \right|_{t=0} + \frac{2\pi S}{|\Gamma|^2} \int_{\Gamma} K(s)W(s)ds .$$

A well known computation yields the result that

$$\frac{\partial}{\partial t} K(s, t) \Big|_{t=0} = - \left( \frac{d^2}{ds^2} W(s) + K(s)^2 W(s) \right) \quad (4.7)$$

Hence in this case, since from (4.3)  $SK(s) - f(s, 0) = S \frac{2\pi}{|\Gamma|}$  we obtain

$$\begin{aligned} \frac{\partial}{\partial t} f(s, 0) + \frac{1}{|\Gamma|} \int_0^{|\Gamma|} f(s, 0) W(s) K(s) ds \\ = S \left( - \frac{d^2}{ds^2} W(s) - K^2(s) W(s) + \frac{1}{|\Gamma|} \int_0^{|\Gamma|} K(s)^2 W(s) ds \right) . \end{aligned} \quad (4.8)$$

We now have what we need to compute the derivative in  $t$ , along the Mullins–Sekerka flow,  $V_0$ , of  $\psi_{V_0}(\cdot, \Gamma_t) \equiv \mu_{0,0}(\cdot, \Gamma_t)$ . We first compute the derivative in  $t$  under a flow generated by  $W$  of  $\psi_V(\cdot, \Gamma_t)$ , see (4.1). Then we set  $V = V_0$  and  $W = V_0$ . A computation just as in the proof of Theorem 4.1 yields

$$\begin{aligned} \frac{\partial}{\partial t} \psi_V(\xi, \Gamma_t) \Big|_{t=0} = \\ \int_{\Gamma} \nabla_{\eta} G(\xi, \eta) \cdot n(\eta) W(\eta) V(\eta, 0) dS_{\eta} \\ + \int_{\Gamma} G(\xi, \eta) \left( \frac{\partial}{\partial t} V(\eta, 0) + K(\eta) W(\eta) V(\eta) \right) dS_{\eta} \\ - \frac{1}{|\Gamma|} \int_{\Gamma} \mathcal{Q}_{\Gamma, W} V(\eta, 0) dS_{\eta} + \frac{|\Gamma'|}{|\Gamma|^2} \int_{\Gamma} \int_{\Gamma} G(\xi, \eta) V(\eta, 0) dS_{\eta} dS_{\xi} \\ - \frac{1}{|\Gamma|} \int_{\Gamma} \int_{\Gamma} G(\xi, \eta) \left( \frac{\partial}{\partial t} V(\eta, 0) + K(\eta) W(\eta) V(\eta, 0) + K(\xi) W(\xi) V(\eta) \right) dS_{\eta} dS_{\xi} . \end{aligned}$$

This simplifies to

$$\begin{aligned} \frac{\partial}{\partial t} \psi_V(\xi, \Gamma_t) \Big|_{t=0} = \int_{\Gamma} \nabla_{\eta} G(\xi, \eta) \cdot n(\eta) W(\eta) V(\eta, 0) dS_{\eta} \\ + \mathcal{E}_{\Gamma, N} \left( \frac{\partial}{\partial t} V(\eta, 0) + K(\eta) W(\eta) V(\eta, 0) \right) - \frac{1}{|\Gamma|} \int_{\Gamma} \mathcal{Q}_{\Gamma, W} V(\eta, 0) dS_{\eta} \\ + \frac{|\Gamma'|}{|\Gamma|^2} \int_{\Gamma} \int_{\Gamma} G(\xi, \eta) V(\eta, 0) dS_{\eta} dS_{\xi} \\ - \frac{1}{|\Gamma|} \int_{\Gamma} \int_{\Gamma} G(\xi, \eta) K(\xi) W(\xi) V(\eta, 0) dS_{\eta} dS_{\xi} . \end{aligned} \quad (4.9)$$

where  $\mathcal{E}_{\Gamma, N}$  is the operator defined in (10.12). From (4.6), arguing as before, the contribution of the last two terms in (4.9) is given by

$$- \frac{1}{|\Gamma|} \int_0^{|\Gamma|} \psi_V(\xi, 0) W(s) K(s) ds . \quad (4.10)$$

From (3.29) and (3.30) when  $\xi \in \Gamma$  and  $V = V_0$ , we have that

$$\begin{aligned} SK(\xi) - \frac{S}{|\Gamma|} \int_{\Gamma} K(s) ds \\ = \int_{\Gamma} G(\xi, \eta) V_0(\eta, 0) dS_{\eta} - \frac{1}{|\Gamma|} \int_{\Gamma} \int_{\Gamma} G(\xi, \eta) V_0(\eta, 0) dS_{\eta} dS_{\xi} = \psi_{V_0}(\xi, \Gamma) . \end{aligned} \quad (4.11)$$

Then (4.10), when  $V = V_0$ , is equal to

$$-S \left( \frac{1}{|\Gamma|} \int_{\Gamma} K^2(\xi) W(\xi) dS_{\xi} - \frac{2\pi}{|\Gamma|^2} \int_{\Gamma} K(\xi) W(\xi) dS_{\xi} \right).$$

Now using (4.8), Theorem 4.1 and the definition of  $\mathcal{E}_{\Gamma, D}$ , see (10.13), we have

$$\begin{aligned} \left. \frac{\partial}{\partial t} \psi_{V_0}(\xi, \Gamma_t) \right|_{t=0} &= \int_{\Gamma} \nabla_{\eta} G(\xi, \eta) \cdot n(\eta) W(\eta) V_0(\eta, 0) dS_{\eta} - \mathcal{E}_{\Gamma, D}(\mathcal{Q}_{\Gamma, W} V_0(\eta)) \\ &+ \mathcal{E}_{\Gamma, N} \left( S\mathcal{T}_{\Gamma} \left( -\frac{d^2}{ds^2} W(s) - K(s)^2 W(s) + \frac{1}{|\Gamma|} \int_0^{|\Gamma|} K(s)^2 W(s) ds \right) \right) \\ &- S \left( \frac{1}{|\Gamma|} \int_{\Gamma} K^2(\xi) W(\xi) dS_{\xi} - \frac{2\pi}{|\Gamma|^2} \int_{\Gamma} K(\xi) W(\xi) dS_{\xi} \right) \\ &= \int_{\Gamma} \nabla_{\eta} G(\xi, \eta) \cdot n(\eta) W(\eta) V_0(\eta, 0) dS_{\eta} - \mathcal{E}_{\Gamma, D}(\mathcal{Q}_{\Gamma, W} V_0(\eta, 0)) \\ &+ S\mathcal{E}_{\Gamma, D} \left( -\frac{d^2}{ds^2} W(s) - K(s)^2 W(s) \right) + \frac{2S\pi}{|\Gamma|^2} \int_{\Gamma} K(\xi) W(\xi) dS_{\xi}. \end{aligned} \quad (4.12)$$

The first term on the right is a double layer potential. Setting  $W = V_0$  where  $V_0$  is the Mullins Sekerka flow in (4.12) we obtain  $D_{V_0} \mu_{0,0}$  which appears in (1.12) and (1.14).

## 5. Results for general $N$

We follow the scheme outlined in the previous sections. We start by amending our ansatz for constructing the approximate solutions by further specifying the nature of the functions  $m_j$ .

**Ansatz – part two:** *Let any number  $\lambda_0 > 0$  be given. For any  $\Gamma \in \mathcal{M}$  with  $\kappa(\Gamma) < 1/(2\lambda_0)$ , let  $m^{(N)}(\cdot, \Gamma) \in C^{\infty}(\Omega)$  be*

$$m^{(N)}(\xi, \Gamma) = m_0 \left( \frac{d(\xi, \Gamma)}{\lambda} \right) + \sum_{j=1}^N \lambda^j m_j(\xi, \Gamma). \quad (5.1)$$

Here, the function  $m_0$  is defined in (3.8). For  $j \geq 1$ , set

$$m_j(\xi, \Gamma) = h_j \left( \frac{d(\xi, \Gamma)}{\lambda}, s(\xi, \Gamma) \right) + \phi_j(\xi, \Gamma) \quad \xi \in \Omega, \quad j = 1, \dots, N. \quad (5.2)$$

Let  $h_j(\cdot, \Gamma)$  be a  $C^{\infty}(\Omega)$  function of the type (2.9). The  $\phi_j$ ,  $j = 1, \dots, N$  satisfy Neuman boundary conditions on  $\partial\Omega$  and a global Lipschitz bound  $\lambda$ -independent, i.e.

$$\|\phi_j\|_{Lip(\Omega)} \leq C \quad j = 1, \dots, N, \quad (5.3)$$

where  $C$  is a constant independent on  $\lambda$ .

**Notational convention:** In the following we denote by  $m^{(N)}(\xi, t) \equiv m^{(N)}(\xi, \Gamma_t^{(N)})$  the function having the requirements prescribed in the ansatz and evaluated at  $\Gamma_t^{(N)}$ ,  $t \in [0, T]$ , the solution of (1.5), being  $T$  its lifetime, see (2.6). We will fix once for all a small value of  $\lambda_0 > 0$ , and define  $\kappa_0 = \frac{1}{2\lambda_0}$ .

This is the upper bound on the curvature that will be tolerated in our estimates, since they suppose

that the local coordinate system introduced in Section 2 is valid for all  $|z| < \lambda_0/\lambda$ . Hence we use this value of  $\kappa_0$  in defining the lifetime  $T$  of our solution of (2.5); see (2.6). We will write

$$m_j(\xi, t) = h_j\left(\frac{d(\xi, \Gamma_t^{(N)})}{\lambda}, \xi, t\right) + \phi_j(\xi, t) \quad j = 1, \dots, N,$$

whenever we need to stress that  $h_j$  depends on  $\Gamma_t^{(N)}$  through the fast scale  $\frac{d(\xi, \Gamma_t^{(N)})}{\lambda}$ . Further we drop in the following the superscript  $(N)$  in  $\Gamma_t^{(N)}$ , writing  $\Gamma_t$ . Through what follows, we write  $C$  to designate a generic positive constant independent on  $\lambda$ . Its actual numerical value may change from one occurrence to the next.

Let  $V_j$ ,  $j = 0, \dots, (N - 1)$  be the sequence of vector fields introduced in the ansatz. We split them, according to (2.7) and (2.8), as

$$V_j = V_j^{(0)} + \langle V_j \rangle \quad j = 0, \dots, N - 1. \quad (5.4)$$

The  $V_j^{(0)}$  will be determined applying the Dirichlet-Neuman operator, by potential theory, in Theorem 5.2. The  $\langle V_j \rangle$ , the part constant on  $\Gamma$ , will be determined in Theorem 5.1, stated next.

**Theorem 5.1** *Fix  $N > 1$ . Let  $\Gamma_t^{(N)}$ ,  $t \in [0, T]$ , be the solution of (1.5) in  $\mathcal{M}$ , being  $T$  its lifetime. Let  $m^{(N)}(\cdot, \Gamma_t^{(N)})$  be as in the ansatz. There is an unique way to determine the  $\langle V_j \rangle$ ,  $j = 0, \dots, (N - 1)$ , such that there exists an unique (up to constant in  $\xi$ ) expansion*

$$\mu^{(N-1)}(\xi, t) = \sum_{i=0}^{N-1} \lambda^i \mu_i(\xi, t) \quad \text{in } \Omega \times [0, T], \quad (5.5)$$

with  $\mu_i$ , for  $i = 0, 1, \dots, N - 1$  given in (6.13), (6.17) and (6.27), satisfying

$$\frac{\partial}{\partial t} m^{(N)}(\xi, t) = \Delta \mu^{(N-1)}(\xi, t) + R_1(\xi, t, \lambda) \quad \text{in } \Omega \times (0, T), \quad (5.6)$$

with  $R_1$  given in (6.9). Further  $\mu^{(N-1)}(\cdot, t)$ , for  $t \in [0, T]$ , is a  $C^\infty(\Omega)$  function satisfying Neumann homogeneous boundary conditions on  $\partial\Omega$ ,

$$\sup_{\xi, t \in \Omega \times [0, T]} |R_1(\xi, t, \lambda)| \leq C(T) \lambda^{N-1} \quad (5.7)$$

and

$$\sup_{t \in [0, T]} \int_{\Omega} |R_1(\xi, t, \lambda)| d\xi \leq C(T) \lambda^N \quad (5.8)$$

where  $C(T)$  is a constant independent on  $\lambda$ .

The proof of Theorems 5.1 is deferred to Section 6. The next theorem assures the existence and (essential) uniqueness of the functions  $m_j$ ,  $j = 0, \dots, N$ , having the properties required in the ansatz. Existence and unicity are obtained provided a compatibility condition is satisfied. This determines  $V_j^{(0)}$ , the orthogonal part of the velocity fields.

**Theorem 5.2** *Let  $T$  be the lifetime of the solution of (1.5) in  $\mathcal{M}$ . Let  $\mu^{(N-1)}(\cdot, t)$ ,  $t \in [0, T]$ , as in Theorem 5.1. Then, provided the  $V_j^{(0)}$ ,  $j = 0, \dots, (N-1)$  are chosen as in (8.18), (8.25) and (8.37), there exist  $m_j$ ,  $j = 0, \dots, N$  having the requirement stated in the ansatz such that*

$$\mu^{(N-1)}(\xi, t) = -\lambda \Delta m^{(N)}(\xi, t) + \frac{1}{\lambda} f(m^{(N)}(\xi, t)) + R_2(\xi, t, \lambda) \quad \text{in } \Omega \times (0, T], \quad (5.9)$$

with  $R_2$  given in (8.45). Further  $m^{(N)}(\cdot, t)$ , for  $t \in [0, T]$ , is a  $C^\infty(\Omega)$  function that satisfies homogeneous Neumann boundary conditions and

$$\sup_{\xi \in \Omega} \sup_{t \in [0, T]} |R_2(\xi, t, \lambda)| \leq C \lambda^N. \quad (5.10)$$

The proof of this result is given in Section 8. Theorem 5.1 and Theorem 5.2 provide the two steps to construct the approximate solution to (1.2). From Theorem 5.1 and 5.2 one obtains easily the following comprehensive result, which amplifies Theorem 1.1. Its proof is given in Section 9.

**Theorem 5.3** *For all  $N > 1$ , there are uniquely defined sequences of vector fields  $V_j$ ,  $j = 0, \dots, (N-1)$ , on  $\mathcal{M}$  and functions  $m_j$ ,  $j = 0, \dots, N$ , from  $\mathcal{M}$  to  $C^\infty(\Omega)$  as in the ansatz such that the following holds. For any  $\Gamma_0 \in \mathcal{M}$ , choose  $k_0 \geq \kappa(\Gamma_0)$ , set  $\lambda_0 = \frac{1}{2k_0}$  and let  $T$  be the lifetime of the solution of (1.5) in  $\mathcal{M}$ , according to (2.6). Then for all  $t < T$ , for all  $\lambda \in (0, \lambda_0]$  we can construct  $(\tilde{m}^{(N)}, \tilde{\mu}^{(N-1)}) \in C^\infty(\Omega \times [0, T])$  where  $\tilde{m}^{(N)}$  is a  $\lambda^N$  modification of  $m^{(N)}$ , i.e.  $\sup_{(\xi, t) \in \Omega \times [0, T]} |\tilde{m}^{(N)}(\xi, t) - m^{(N)}(\xi, t)| \leq C \lambda^N$  and  $\tilde{\mu}^{(N-1)}$  is a  $\lambda^{N-1}$  modification of  $\mu^{(N-1)}$ , i.e.  $\sup_{(\xi, t) \in \Omega \times [0, T]} |\tilde{\mu}^{(N-1)}(\xi, t) - \mu^{(N-1)}(\xi, t)| \leq C \lambda^{N-1}$  satisfying*

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{m}^{(N)}(\xi, t) &= \Delta \tilde{\mu}^{(N-1)}(\xi, t) \quad \text{in } \Omega \times (0, T) \\ \tilde{\mu}^{(N-1)}(\xi, t) &= -\lambda \Delta \tilde{m}^{(N)}(\xi, t) + \frac{1}{\lambda} f(\tilde{m}^{(N)}(\xi, t)) + R(\xi, t, \lambda) \quad \text{in } \Omega \times (0, T). \end{aligned} \quad (5.11)$$

where

$$\sup_{\xi \in \Omega} \sup_{t \in [0, T]} |R(\xi, t, \lambda)| \leq C \lambda^{N-1}.$$

Further,  $\tilde{\mu}^{(N-1)}(\cdot, t)$  and  $\tilde{m}^{(N)}(\cdot, t)$ , for  $t \in [0, T]$ , satisfy Neumann homogeneous boundary conditions on the boundary of  $\Omega$ . In addition

$$\sup_{t \in [0, T]} \sup_{\xi \in \Omega} |\tilde{\mu}^{(N-1)}(\xi, t) - \mu_{0,0}(\xi, t)| \leq C \lambda, \quad (5.12)$$

where  $\mu_{0,0}$  is the solution of (1.8), (1.9),

$$\sup_{t \in [0, T]} \sup_{\xi \in \mathcal{N}(\lambda_0, \Gamma_t^{(N)})} \left| \tilde{m}^{(N)}(\xi, t) - \bar{m} \left( \frac{d(\xi, \Gamma_t^{(N)})}{\lambda} \right) \right| \leq C \lambda, \quad (5.13)$$

$$\sup_{t \in [0, T]} \sup_{\xi \in \Omega \setminus \mathcal{N}(\frac{\lambda_0}{2}, \Gamma_t^{(N)})} |\tilde{m}^{(N)}(\xi, t) \mp 1| \leq C \lambda. \quad (5.14)$$

Once the approximate solution to (1.2) is constructed it remains to show that it is indeed ‘‘close’’ to the solution of (1.2). This can be achieved arguing as in [1]. One needs only to verify that the spectral

estimate used there still holds for the approximated solution we constructed. This is indeed the case, as can be easily verified.

## 6. Construction of the approximate chemical potential

In this section we apply classical potential theory to prove Theorem 5.1. We look for a function  $\mu^{(N-1)}$  from  $\mathcal{M}$  to  $C^\infty(\Omega)$  having the form

$$\mu^{(N-1)}(\xi, \Gamma) = \sum_{i=0}^{N-1} \lambda^i \mu_i(\xi, \Gamma) \quad \xi \in \Omega, \quad (6.1)$$

where  $\mu_i$ ,  $i = 0, \dots, N-1$ , are functions to be determined. We insert  $m^{(N)}$ , as in the ansatz, and  $\mu^{(N-1)}$ , as in (6.1), both evaluated at  $\Gamma_t^{(N)}$  where  $\Gamma_t^{(N)}$  is the solution of (1.5), into (3.1). We obtain  $(N-1)$  Laplace equations for  $\mu_i(\cdot, \Gamma_t^{(N)})$ ,  $i = 1, \dots, (N-1)$ . The compatibility condition needed to solve these Laplace equations determines  $\langle V_j \rangle(\Gamma_t^{(N)})$  for  $j = 0, \dots, (N-1)$ .

When differentiating  $m^{(N)}(\cdot, \Gamma_t^{(N)})$  with respect to  $t$  we need to take into account that  $m^{(N)}$  depends on  $\Gamma_t$ , through a fast and slow scale, the fast scale brings a factor  $\lambda^{-1}$ .

**Notation** Let  $m$  be a function from  $\mathcal{M}$  to  $C^\infty(\Omega)$  of the type (2.9). Let  $V$  be a vector field on  $\mathcal{M}$ . We denote by

$$D_V m(\xi, \Gamma) = \frac{1}{\lambda} h' \left( \frac{d(\xi, \Gamma)}{\lambda}, s(\xi, \Gamma) \right) V(s(\xi)), \quad (6.2)$$

where we indicate with prime the derivative of  $h$  with respect to the first variable  $z = \frac{d(\xi, \Gamma)}{\lambda}$ . When  $W_N = \sum_{j=0}^{N-1} \lambda^j V_j$ , with  $V_0, \dots, V_{N-1}$  a vector fields on  $\mathcal{M}$

$$D_{W_N} m(\xi, \Gamma) = \sum_{j=0}^{N-1} \lambda^j D_{V_j} m(\xi, \Gamma).$$

Note that by the orthogonality of  $\nabla_\xi d$  with respect to the surface there is no contribution in (6.2) from  $s(\xi, \Gamma)$ . We have then

$$\begin{aligned} \frac{\partial m^{(N)}}{\partial t}(\xi, t) &= D_{W_N}(m^{(N)}) = D_{V_0} m_0 + \lambda [D_{V_1} m_0 + D_{V_0} m_1] \\ &+ \lambda^2 [D_{V_1} m_1 + D_{V_0} m_2 + D_{V_2} m_0] + \dots + \lambda^{N-1} \left[ \sum_{i=0}^{N-1} D_{V_i} m_{N-1-i} \right] + R_N + E \end{aligned} \quad (6.3)$$

where

$$R_N \equiv \lambda^N \left[ \sum_{i=0}^{N-1} D_{V_i} m_{N-i} \right] + \mathcal{O}(\lambda^N). \quad (6.4)$$

The term  $E \equiv E(\xi, t, \lambda)$  is obtained deriving  $r\left(\frac{d(\xi, \Gamma_t)}{\lambda_0}\right)$  with respect to the velocity field the function

$$E(\xi, t, \lambda) = \frac{1}{\lambda_0} r' \left( \frac{d(\xi, \Gamma_t)}{\lambda_0} \right) \left[ \sum_{i=0}^{N-1} \lambda^i V_i(\sigma(\xi), t) \right] \left\{ \bar{m} - [\mathbb{I}_{\{d(\xi, \Gamma_t) > 0\}} - \mathbb{I}_{\{d(\xi, \Gamma_t) < 0\}}] \right\} \quad (6.5)$$

It is exponentially small, namely  $r'$  is different from zero only for  $\frac{\lambda_0}{2\lambda} \leq |z| \leq \frac{\lambda_0}{\lambda}$  and  $\bar{m}$  goes exponentially to  $\pm 1$ . Taking into account (6.3) and (3.1) we obtain a set of  $N$  equations for the  $\mu_i$ ,  $i = 0, \dots, N-1$ .

**Zero order term in  $\lambda$**

$$D_{V_0} m_0 \equiv \frac{1}{\lambda} V_0 m' = \Delta \mu_0 \quad \text{for } \xi \in \Omega, \quad (6.6)$$

**First order term in  $\lambda$**

$$[D_{V_1} m_0 + D_{V_0} m_1] = \Delta \mu_1 \quad \text{for } \xi \in \Omega \quad (6.7)$$

**$n$ -th order term in  $\lambda$  ( $n \leq N-1$ )**

$$\left[ \sum_{i=0}^n D_{V_i} m_{n-i} \right] = \Delta \mu_n \quad \text{for } \xi \in \Omega. \quad (6.8)$$

**Remainder term**

The remainder term, see (6.4) and (6.5) is given by

$$R_1(\xi, t, \lambda) = R_N(\xi, t) + E(\xi, t, \lambda) \quad (6.9)$$

It can be easily estimated

$$\sup_{(\xi, t) \in \Omega \times [0, T]} |R_1(\xi, t)| \leq C(T) \lambda^{N-1}. \quad (6.10)$$

Further, one gains an extra power of  $\lambda$  when integrating  $R_1$ , since the terms of order  $\lambda^{N-1}$  have support in  $\mathcal{N}(\lambda_0)$ ,

$$\sup_{t \in [0, T]} \int_{\Omega} |R_1(\xi, t)| d\xi \leq C(T) \lambda^N. \quad (6.11)$$

Next we show existence and uniqueness (up to constant) of the solutions of the equations obtained at different order. In Lemma 6.1 and in Lemma 6.2 we consider respectively the first and second order equation, since for  $d = 2$  the first order term, does not require the extra device we need for higher order terms. These equations were already discussed in Section 3. We repeat here to make the presentation more systematic. Finally in Lemma 6.3 we outline the proof for solving the equation to a generic order.

**Remark:** In the next lemmas, as well in all the paper,  $G$  stands for the Neumann Green's function on  $\Omega$ , see (10.1).

**Lemma 6.1** *There exists an unique (up to constant in  $\xi$ ) solution of (6.6) provided*

$$\int_{\Gamma_t} V_0(\eta, \Gamma_t) dS_\eta = 0 \quad t \in [0, T]. \quad (6.12)$$

*It is given by*

$$\mu_0(\xi, \Gamma_t) = \int_{\Omega} G(\xi, \eta) \left( \frac{1}{\lambda} m'_0 \left( \frac{d(\eta, \Gamma_t)}{\lambda} \right) V_0(s(\eta), t) \right) d\eta + c_0(t), \quad (6.13)$$

where  $c_0(t)$  is a constant (in  $\xi$ ) to be determined. It is a  $C^\infty(\Omega)$  function for  $t \in [0, T]$ .

**Proof:** The solvability of (6.6) requires that for all  $t \in [0, T]$

$$\int_{\Omega} \left( \frac{1}{\lambda} m'_0 \left( \frac{d(\eta, \Gamma_t)}{\lambda} \right) V_0(s(\eta), t) \right) d\eta = 0 \quad (6.14)$$

This forces to take  $V_0$  such that (6.12) holds. Now since

$$d\eta = \lambda(1 - \lambda z K(s)) ds dz \quad (6.15)$$

and  $m'_0$  is even we have that (6.14) is satisfied.  $\square$

**Remark** In dimension  $d = 2$ , the velocity field  $V_0$  coincides with  $V_0^{(0)}$ , the constant part being zero. If we were working in three or more dimensions, the integral (6.14) would not have vanished identically, but would have been a term of  $\mathcal{O}(\lambda^2)$ . This would have caused only a slight complication, and we shall explain how to deal with such problems in Lemma 6.2 when we discuss the first order term.

**Lemma 6.2** *There exists a unique (up to constant in  $\xi$ ) solution of (6.7) provided*

$$V_1(\Gamma_t) \equiv V_1^{(0)}(\Gamma_t) + \langle V_1 \rangle(\Gamma_t)$$

with

$$\int_{\Gamma_t} V_1^{(0)}(\eta, \Gamma_t) dS_\eta = 0 \quad \forall t \in [0, T] \quad (6.16)$$

and  $\langle V_1 \rangle$  chosen according to (6.23). It is given by

$$\mu_1(\xi, t) = \mu_{1,0}(\xi, t) + \tilde{\mu}_1(\xi, t) \quad (6.17)$$

where  $\tilde{\mu}_1$  is given in (6.26),

$$\mu_{1,0}(\xi, t) = \int_{\Omega} G(\xi, \eta) \left( \frac{1}{\lambda} m'_0 \left( \frac{d(\eta, \Gamma_t)}{\lambda} \right) V_1^{(0)}(s(\eta), t) \right) d\eta + c_1(t), \quad (6.18)$$

and  $c_1(t)$  is a constant (in  $\xi$ ) to be determined. The solution is a  $C^\infty(\Omega)$  function for  $t \in (0, T]$ .

**Proof:** The solvability of (6.7) requires

$$\int_{\Omega} [D_{V_1} m_0 + D_{V_0} m_1] d\xi = 0 \quad (6.19)$$

for any  $t \in [0, T]$ . Here we are assuming that  $m_1$ ,  $m_0$  and  $V_0$  are already determined and so we define

$$b_1(t) = \int_{\Omega} D_{V_0} m_1 d\xi. \quad (6.20)$$

Set

$$V_1(\Gamma_t) \equiv V_1^{(0)}(\Gamma_t) + \langle V_1 \rangle(\Gamma_t) \quad (6.21)$$

Require (6.16). Then we obtain

$$\int_{\Omega} D_{V_1} m_0 d\xi = 2|\Gamma_t| \langle V_1 \rangle(\Gamma_t). \quad (6.22)$$

Hence, to satisfy (6.19) we must take

$$\langle V_1(\Gamma_t) \rangle = -\frac{1}{2|\Gamma_t|} b_1(t). \quad (6.23)$$

This determines  $\langle V_1(\Gamma_t) \rangle$ , the projection of  $V_1(\Gamma_t)$  onto the constants. It still remains to determine the orthogonal part  $V_1^{(0)}$ . The solution of (6.7) exists and it is given by

$$\mu_1(\xi, t) = \int_{\Omega} G(\xi, \eta) [D_{V_1} m_0 + D_{V_0} m_1] d\eta + c_1(t) \quad (6.24)$$

Since decomposition (6.21) and for further use, it is convenient to write

$$\mu_1(\xi, t) = \mu_{1,0}(\xi, t) + \tilde{\mu}_1(\xi, t) \quad (6.25)$$

where  $\mu_{1,0}(\xi, t)$  is given in (6.18) and

$$\tilde{\mu}_1(\xi, t) = \int_{\Omega} G(\xi, \eta) D_{V_0} m_1 d\eta + \langle V_1(t) \rangle \int_{\Omega} G(\xi, \eta) \left( \frac{1}{\lambda} m'_0 \left( \frac{d(\eta, \Gamma_t)}{\lambda} \right) \right) d\eta \quad (6.26)$$

□

**Lemma 6.3** *The solution of (6.8), for  $2 \leq j \leq N - 1$  exists and is unique (up to constant in  $\xi$ ) provided*

$$V_j(\Gamma_t) \equiv V_j^{(0)}(\Gamma_t) + \langle V_j(\Gamma_t) \rangle,$$

$$\int_{\Gamma_t} V_j^{(0)}(s, \Gamma_t) ds = 0 \quad \forall t \in [0, T]$$

and  $\langle V_j(\Gamma_t) \rangle$  chosen according to (6.34). It is given by

$$\mu_j(\xi, t) = \mu_{j,0}(\xi, t) + \tilde{\mu}_j(\xi, t) \quad (6.27)$$

where

$$\mu_{j,0}(\xi, t) = \int_{\Omega} G(\xi, \eta) \left( \frac{1}{\lambda} m'_0 \left( \frac{d(\eta, \Gamma_t)}{\lambda} \right) \right) V_j^{(0)}(s(\eta), t) d\eta + c_j(t) \quad (6.28)$$

and  $\tilde{\mu}_j$  is

$$\begin{aligned} \tilde{\mu}_j(\xi, t) &= \int_{\Omega} G(\xi, \eta) \left[ \sum_{n=0}^{j-1} D_{V_n} m_{j-n} \right] d\eta \\ &+ \langle V_j(\Gamma_t) \rangle \int_{\Omega} G(\xi, \eta) \left( \frac{1}{\lambda} m'_0 \left( \frac{d(\eta, \Gamma_t)}{\lambda} \right) \right) d\eta \end{aligned} \quad (6.29)$$

The solution  $\mu_j(\cdot, t)$ , for  $t \in (0, T]$  is a  $C^\infty(\Omega)$  function.

**Proof:** The proof goes as in Lemma 6.2. The solution exists if

$$\int_{\Omega} \left[ \sum_{n=0}^j D_{V_n} m_{j-n} \right] d\xi = 0 \quad (6.30)$$

for any  $t \in [0, T]$ . Here,  $D_{V_n} m_{j-n}$  for  $n = 0, \dots, j-1$  are determined and so we define

$$b_j(t) = \int_{\Omega} \left[ \sum_{n=0}^{j-1} D_{V_n} m_{j-n} \right] d\xi . \quad (6.31)$$

Requiring

$$V_j(\Gamma_t) \equiv V_j^{(0)}(\Gamma_t) + \langle V_j(\Gamma_t) \rangle$$

with

$$\int_{\Gamma} V_j^{(0)}(s, \Gamma_t) ds = 0 . \quad (6.32)$$

gives

$$\int_{\Omega} D_{V_j} m_0 d\xi = 2|\Gamma_t| \langle V_j(\Gamma_t) \rangle . \quad (6.33)$$

Hence to fulfill (6.30) we must take

$$\langle V_j(\Gamma_t) \rangle = -\frac{1}{2|\Gamma_t|} b_j(t) . \quad (6.34)$$

This determines  $\langle V_j(\Gamma_t) \rangle$ , the projection of  $V_j(\Gamma_t)$  onto the constants. It still remains to determine the orthogonal part  $V_j^{(0)}$ . The solution of (6.8) exists and, as done before, is represented by (6.27).  $\square$

### Proof of Theorem 5.1

From Lemma 6.1, Lemma 6.2 and Lemma 6.3 we have that  $\mu^{(N-1)}$ , satisfies by construction (1.1). The remainder  $R_1$  is defined in (6.9) and estimated in (6.10) and (6.11). The  $\mu^{(N-1)}(\cdot, t)$  for  $t \in [0, T]$  satisfies homogeneous Neumann boundary conditions by construction. Theorem 5.1 is then proved.  $\square$

## 7. Proof of Theorem 5.2: Derivation of the equations

In this section we begin the proof of Theorem 5.2. We write (3.2), inserting in the left side the function  $\mu^{(N-1)}(\cdot, \Gamma_t^{(N)})$  determined in Theorem 5.1, see (5.5):

$$\lambda \mu^{(N-1)}(\xi, t) = -\lambda^2 \Delta m(\xi, t) + f(m(\xi, t)) \quad \text{in } \Omega \times (0, T) \quad (7.1)$$

The  $\mu^{(N-1)}$  are written in terms of the  $m^{(N)}$ , chosen according to the ansatz. Here we prove that there exists an unique way to find the function  $m^{(N)}$ , having indeed the property required in the ansatz and satisfying equation (7.1) in the sense of Theorem 5.2. The existence at any order of the  $m_j$ ,  $j = 0, \dots, N$  is obtained provided a compatibility condition is satisfied. This compatibility condition imposes to take  $V_j^{(0)}$ ,  $j = 0, 1, \dots, (N-1)$ , according to (8.9), (8.25) and (8.37). In the proof of the Theorem 5.2 we distinguish two main steps

- **step 1:** *Determination at any order of the equations. This is carried out in this section.*
- **step 2:** *Analysis of the equations derived in the first step. This will be done in the next section.*

In this and in the next section  $\Gamma_t$  is kept fixed, so to simplify notations we drop to write the dependence on  $t$ , writing explicitly only when we think it is worth to remind it.

To separate the fast and slow scale of  $m^{(N)}$  near the surface  $\Gamma$ , we write the Laplacian in the system of local coordinates introduced in Subsection 2.1. The expansion in  $\lambda$  of the Laplacian written in this

coordinate system is reported in the appendix. We match the right and left terms of the equations having the same power of  $\lambda$ , distinguishing the case where  $\xi \in \mathcal{N}(\lambda_0)$  from the one with  $\xi \in \Omega \setminus \mathcal{N}(\lambda_0)$ . We therefore get at any order two sets of equations, one for  $\xi \in \mathcal{N}(\lambda_0)$  and the other for  $\xi \in \Omega \setminus \mathcal{N}(\lambda_0)$ . After simple, however lengthly computations we obtain the following. Taking into account a formula from the appendix, namely (10.15), and denoting by  $'$  the derivative with respect to  $z$ , and letting  $a_n, b_n, c_n$  denote the quantities defined in (10.16),

$$\begin{aligned} \lambda^2 \Delta m^{(N)}(z, s) &= \left\{ \bar{m}''(z) + \sum_{n=1}^N \lambda^n [h_n''(z, s) + a_n(z, s) \bar{m}'] \right\} \\ &+ \left\{ \sum_{n=2}^N \lambda^n \sum_{i=1}^{n-1} a_{n-i}(z, s) h_i'(z, s) + \sum_{n=3}^N \lambda^n \left[ \sum_{i=1}^{n-2} b_{n-i}(z, s) \frac{d^2}{ds^2} h_i(z, s) \right] \right\} \\ &+ \left\{ \sum_{n=4}^N \lambda^n \sum_{i=1}^{n-3} c_{n-i}(z, s) \frac{d}{ds} h_i(z, s) \right\} + \lambda^2 \Delta \left[ \sum_{i=1}^N \lambda^i \phi_i(\xi) \right] + E_1(\xi, t, \lambda) + \lambda^{N+1} A(\xi, t, \lambda) \end{aligned} \quad (7.2)$$

with

$$\sup_{(\xi, t) \in \Omega \times [0, T]} |A(\xi, t, \lambda)| \leq C(T), \quad (7.3)$$

$$\sup_{t \in [0, T]} \int_{\Omega} d\xi |A(\xi, t, \lambda)| \leq \lambda C(T), \quad (7.4)$$

$$\begin{aligned} E_1(\xi, \lambda) &\equiv \lambda^2 \Delta r \left( \frac{d(\xi, \Gamma)}{\lambda_0} \right) \left\{ \bar{m} \left( \frac{d(\xi, \Gamma)}{\lambda} \right) - [\mathbb{1}_{\{d(\xi, \Gamma) > 0\}} - \mathbb{1}_{\{d(\xi, \Gamma) < 0\}}] \right\} \\ &+ 2\lambda^2 \nabla r \cdot \nabla \left[ \bar{m} \left( \frac{d(\xi, \Gamma)}{\lambda} \right) \right], \end{aligned}$$

and

$$\lim_{\lambda \rightarrow 0} \sup_{(\xi, t) \in \Omega \times [0, T]} |E_1(\xi, t, \lambda)| = 0 \quad (7.5)$$

the convergence being exponentially fast due to the decay of  $\bar{m}$ .

Define  $f_i$  such that

$$\begin{aligned} f(m^{(N)}) &= f(m_0) + f'(m_0) \left[ \sum_{i=1}^N \lambda^i m_i \right] + \sum_{i=2}^N \lambda^i f_i(m_0, m_1, \dots, m_{i-1}) \\ &+ \lambda^{N+1} B_{N+1}(\cdot, \lambda) \end{aligned} \quad (7.6)$$

$$\sup_{\xi \in \Omega, t \in [0, T]} |B_{N+1}(\xi, t, \lambda)| \leq C \quad (7.7)$$

One easily obtains the  $f_i$ , for  $i = 2, \dots, N$  Taylor expanding up to  $N$ - order  $f$  around  $m_0$  and collecting terms having the same power of  $\lambda$ . We insert (7.2) and (7.6) into (7.1). We equate terms having the same order (when estimated with the  $L^\infty(\Omega)$  norm) in  $\lambda$  obtaining at any order two equations one for  $\xi \in \Omega \setminus \mathcal{N}(\lambda_0)$ , the other for  $\xi \in \mathcal{N}(\lambda_0)$ . The one for  $\xi \in \Omega \setminus \mathcal{N}(\lambda_0)$ , determines the  $\phi_i$ , the slowly changing terms, the other for  $\xi \in \mathcal{N}(\lambda_0)$  determines the  $h_i$ , the rapidly decaying terms. When deriving the equations for  $\xi \in \mathcal{N}(\lambda_0)$  terms of the type  $\Delta \phi_i(\lambda z, s)$  appear. As was shown for the first term in the expansion, see Subsection 3.1, the  $\phi_i$  are  $C^\infty$  functions, since are proportional to the  $\mu_i$ , and have

the same type of singularity in  $\lambda$  when differentiated in  $\xi$ . Therefore terms  $\lambda^{n+1}\Delta\phi_{n-1}(\lambda z, s) \equiv \mathcal{O}(\lambda^n)$  and we write them in the  $\lambda^n$  order equation.

**Zero order term in  $\lambda$**

$$0 = -r\left(\frac{d(\xi, \Gamma)}{\lambda_0}\right)m_0''(z) + f(m_0(z)) \quad \text{for} \quad z \in \left[-\frac{\lambda_0}{\lambda}, \frac{\lambda_0}{\lambda}\right] \quad (7.8)$$

$$f(\pm 1) = 0 \quad \text{for} \quad \xi \in \Omega \setminus \mathcal{N}(\lambda_0) \quad (7.9)$$

**First order term in  $\lambda$ :**

$$\begin{aligned} \mu_0(\lambda z, s) = & -[h_1''(z, s) - K(s)m_0'(z)] \\ & + f'(m_0)[h_1(z, s) + \phi_1(\lambda z, s)] \quad \text{for} \quad \xi \in \mathcal{N}(\lambda_0) \end{aligned} \quad (7.10)$$

and

$$\mu_0(\xi) = f'(1)\phi_1(\xi) \quad \text{for} \quad \xi \in \Omega \setminus \mathcal{N}(\lambda_0) \quad (7.11)$$

**Second order term in  $\lambda$ :**

$$\begin{aligned} \mu_1(\lambda z, s) = & -[h_2''(z, s) - K^2(s)zm_0'(z) - K(s)h_1'(z, s)] \\ & + f'(m_0(z))[h_2(z, s) + \phi_2(\lambda z, s)] - \lambda\Delta\phi_1(\lambda z, s) + f_2(m_0, m_1)(\lambda z, s) \end{aligned} \quad (7.12)$$

for  $\xi \in \mathcal{N}(\lambda_0)$ .

$$\mu_1(\xi) = f'(1)\phi_2(\xi) + f_2(1, \phi_1(\xi)) - \lambda\Delta\phi_1(\xi) \quad \text{for} \quad \xi \in \Omega \setminus \mathcal{N}(\lambda_0) \quad (7.13)$$

More explicitly the  $f_2$  term is given

$$f_2(m_0, m_1)(\lambda z, s) = \frac{1}{2}f''(m_0(z))[h_1^2(z, s) + \phi_1^2(\lambda z, s) + 2\phi_1(\lambda z, s)h_1(z, s)]$$

for  $\xi \in \mathcal{N}(\lambda_0)$ , and

$$f_2(1, \phi_1(\xi)) = \frac{1}{2}f''(1)\phi_1^2(\xi)$$

for  $\xi \in \Omega \setminus \mathcal{N}(\lambda_0)$ .

**n-th order term in  $\lambda$  ( $3 \leq n \leq N$ ):**

$$\begin{aligned} & \mu_{n-1}(\lambda z, s) \\ = & - \left[ h_n''(z, s) + a_n(z, s)m_0'(z) + \sum_{i=1}^{n-1} [a_{n-i}(z, s)h_i'(z, s)] + \sum_{i=1}^{n-2} b_{n-i}(z, s)\frac{d^2}{ds^2}h_i(z, s) \right. \\ & \left. + \mathbb{I}_{\{n \geq 4\}} \sum_{i=1}^{n-3} c_{n-i}(z, s)\frac{d}{ds}h_i(z, s) \right] - \lambda\Delta\phi_{n-1}(\lambda z, s) + f'(m_0)[h_n(z, s) + \phi_n(\lambda z, s)] \\ & + f_n(m_0, m_1, m_2, \dots, m_{n-1})(\lambda z, s) \quad \xi \in \mathcal{N}(\lambda_0) \end{aligned} \quad (7.14)$$

$$\mu_{n-1}(\xi) = -\lambda \Delta \phi_{n-1}(\xi) + f'(1)\phi_n(\xi) + f_n(\pm 1, \phi_1, \phi_2, \dots, \phi_{n-1})(\xi) \quad \xi \in \Omega \setminus \mathcal{N}(\lambda_0) \quad (7.15)$$

**The Remainder:**

The remainder term is given by

$$\lambda \tilde{R}_2(\xi, t, \lambda) = \lambda^{N+1} A(\xi, t, \lambda) + \lambda^{N+2} \Delta \phi_N(\xi, t) + E_1(\xi, t, \lambda) + \lambda^{N+1} B_{N+1}(\xi, t, \lambda). \quad (7.16)$$

From (7.3), (7.5), (7.7), we have

$$\sup_{(\xi, t) \in \Omega \times [0, T]} |\tilde{R}_2(\xi, t, \lambda)| \leq C \lambda^N. \quad (7.17)$$

## 8. Proof of Theorem 5.2: Analysis of compatibility conditions

In this section we analyze the equations obtained in Section 7. As in Section 3, the strategy is to find at each order in  $\lambda$ , first, the slowly varying part, the  $\phi_i$ , solving the equations for  $\xi \in \Omega \setminus \mathcal{N}(\lambda_0, \Gamma_t^{(N)})$ . Then we extend  $\phi_i$  globally in  $\Omega$  and determine the rapidly decaying part  $h_i$  solving the equations in  $\xi \in \mathcal{N}(\frac{\lambda_0}{2}, \Gamma_t^{(N)})$ . However here, in order to continue to arbitrary order, it is convenient to modify the way we extract the compatibility condition required to solve the equation for the  $h_i$ . The modification is to add and subtract to each order a term of lower order  $\lambda^{i+1} \alpha_i(s, \Gamma) \bar{m}'(z)$ , with  $\alpha_i(\cdot, \Gamma) \in C^\infty(\Gamma)$ . Adding and subtracting terms does not change, of course, the total quantity but it modifies the equation we obtain at each single order. We obtain at any  $i \geq 1$  order in  $\lambda$  the corresponding  $m_i$  split in one part, the function  $\phi_i$  defined globally in  $\Omega$ , satisfying Neumann condition on the boundary of  $\Omega$ , the other part, the  $h_i$ , is different from zero only in a tubular neighborhood of  $\Gamma$ ,  $\mathcal{N}(\frac{\lambda_0}{2}, \Gamma_t^{(N)})$  and it is exponentially decaying to 0 far from  $\Gamma$ . The 0 order term is different, in the sense that  $m_0$  far from the interfaces relaxes exponentially fast to  $\pm 1$ . We first state the following Lemma, taken from [1]. We use this to determine the condition for solvability of equations of the type (8.1), where  $\mathcal{L}$  is the operator on  $L^2(\mathbb{R})$  defined in (3.23).

**Lemma 8.1** [ABC] *Let  $A(z, s, t)$ ,  $z \in \mathbb{R}$ ,  $s \in \Gamma$ ,  $t \in [0, T]$ . Assume that there exists  $A^\pm(s, t)$  such that for  $A(z, s, t) - A^\pm(s, t) = \mathcal{O}(e^{-\alpha|z|})$  as  $|z| \rightarrow \infty$  for  $s \in \Gamma$  and  $t \in [0, T]$ . Then for each  $s \in \Gamma$  and  $t \in [0, T]$*

$$\begin{aligned} (\mathcal{L}w)(z, s, t) &= A(z, s, t) \quad \text{for} \quad z \in \mathbb{R} \\ w(0, s, t) &= 0, w(\cdot, s, t) \in L^\infty(\mathbb{R}) \end{aligned} \quad (8.1)$$

has a solution if and only if

$$\int_{\mathbb{R}} A(z, s, t) \bar{m}'(z) dz = 0 \quad \text{for all} \quad s \in \Gamma, t \in [0, T] \quad (8.2)$$

In addition if the solution exists, then it is unique and satisfies

$$D_z^\ell \left[ w(z, s, t) + \frac{A^\pm(s, t)}{f'(1)} \right] = \mathcal{O}(e^{-\alpha|z|}) \quad \text{as} \quad |z| \rightarrow \infty \quad \text{and} \quad \ell = 0, 1, 2 \quad (8.3)$$

Furthermore if  $A(z, s, t)$  satisfies

$$D_s^m D_t^n D_z^\ell [A(z, s, t) - A^\pm(s, t)] = \mathcal{O}(e^{-\alpha|z|})$$

then

$$D_s^m D_t^n D_z^\ell \left[ w(z, s, t) + \frac{A^\pm(s, t)}{f'(1)} \right] = \mathcal{O}(e^{-\alpha|z|})$$

for all  $m = 0, 1 \dots M$ ,  $n = 0, 1 \dots N$ , and  $\ell = 0, 1 \dots L + 2$ . Further, since  $\mathcal{L}$  is a preserving parity operator, the solution  $w(z, s, t)$  is odd (even) with respect to  $z$  if  $A(z, s, t)$  is odd (even) with respect to  $z$  for  $s \in \Gamma$  and  $t \in [0, T]$ .

**Remark:** In case  $A(\cdot, \cdot, \cdot) \in C^\infty(\mathbb{R} \times \Gamma \times [0, T])$ , the solution  $w(\cdot, \cdot, \cdot)$  of (8.1) is  $C^\infty(\mathbb{R} \times \Gamma \times [0, T])$ . Whenever we apply Lemma 8.1, the right hand side of (8.7) will be  $C^\infty(\mathbb{R} \times \Gamma \times [0, T])$ , then the solutions will be in  $C^\infty(\mathbb{R} \times \Gamma \times [0, T])$ .

The compatibility conditions must hold for every  $\Gamma$  in  $\mathcal{M}$  and so in our derivation we do must refer to  $\Gamma_t$ .

### Zero order term in $\lambda$ :

For  $\xi \in \mathcal{N}(\frac{\lambda_0}{2})$  we have from (7.8)

$$0 = -\bar{m}''(z) + f(\bar{m}(z)) \quad \text{for} \quad z \in \left[-\frac{\lambda_0}{2\lambda}, \frac{\lambda_0}{2\lambda}\right] \quad (8.4)$$

and from (7.9)

$$0 = f(\pm 1) \quad \text{for} \quad \xi \in \Omega \setminus \mathcal{N}(\lambda_0) \quad (8.5)$$

The (8.4) and (3.7) are satisfied by  $\bar{m}$ , see (3.6) and (3.7).

### First order term in $\lambda$ :

As explained at the beginning of this section, it is convenient for solving (7.10) to add a term  $\lambda\alpha_1(s, \Gamma)\bar{m}'(z)$ ,  $s \in \Gamma$  and  $z \in \mathbb{R}$ , with  $\alpha_1(\cdot, \Gamma)$  to be determined. This term will be subtracted to the second order. In the following we will short notation, writing  $\alpha_1(s) \equiv \alpha_1(s, \Gamma)$ . Recalling the definition of  $\mathcal{L}$ , see (3.23), adding  $\lambda\alpha_1(s)\bar{m}'(z)$ , we write (7.10) as

$$\mu_0(\lambda z, s) - f'(\bar{m}(z))\phi_1(\lambda z, s) - K(s)\bar{m}'(z) + \lambda\alpha_1(s)\bar{m}'(z) = (\mathcal{L}h_1)(z, s) \quad (8.6)$$

for  $\xi \in \mathcal{N}(\frac{\lambda_0}{2})$ . One has from (7.11) that

$$\phi_1(\xi) = \frac{\mu_0(\xi)}{f'(1)} \quad \text{for} \quad \xi \in \Omega \setminus \mathcal{N}(\lambda_0). \quad (8.7)$$

We extend this definition of  $\phi_1$  globally in  $\Omega$ . We then insert (8.7) into (8.6) obtaining for  $s \in \Gamma, |z| \leq \frac{\lambda_0}{2\lambda}$ .

$$\mu_0(\lambda z, s) \left[ 1 - \frac{f'(\bar{m}(z))}{f'(1)} \right] - K(s)\bar{m}'(z) + \lambda\alpha_1(s)\bar{m}'(z) = (\mathcal{L}h_1)(z, s) \quad (8.8)$$

Since the left hand side of (8.8) tends exponentially to 0 as  $z \rightarrow \pm\infty$  if the solution of (8.8) exists, see Lemma 8.1, then decays exponentially fast to 0. We can therefore extend (8.8) for  $z$  in all  $\mathbb{R}$ . We have the following result.

**Lemma 8.2** *Set*

$$V_0^{(0)}(\xi, \Gamma) = S\mathcal{T}_\Gamma \left[ K(\cdot) - \frac{1}{|\Gamma|} \int_\Gamma K(\eta) dS_\eta \right] (\xi) \quad \xi \in \Gamma, \quad (8.9)$$

where

$$S = \frac{1}{4} \int_{\mathbb{R}} (\bar{m}'(z))^2 dz = \frac{\sqrt{2}}{3} \quad (8.10)$$

and  $\mathcal{T}_\Gamma$  is the operator defined in (10.8). Then it is uniquely determined  $\alpha_1(\cdot, \Gamma) \in C^\infty(\Gamma)$  and it exists an unique solution of (8.8),  $h_1(\cdot, s)$ ,  $s \in \Gamma$ , such that  $h_1(0, s) = 0$  and  $h_1(\cdot, s) \in L^\infty(\mathbb{R})$ . Moreover  $h_1(\cdot, s)$ , for  $s \in \Gamma$  is even as function of  $z$  and its derivatives with respect to  $z$  decay exponentially to 0 as  $z$  tends to  $\pm\infty$ .

**Proof:** For any fixed  $s \in \Gamma$ , the condition for the existence of  $h_1$ , see (8.2), requires

$$\begin{aligned} & \int_{\mathbb{R}} \mu_0(\lambda z, s) \left[ 1 - \frac{f'(\bar{m}(z))}{f'(1)} \right] \bar{m}'(z) dz \\ &= [K(s) - \lambda \alpha_1(s)] \int_{\mathbb{R}} (\bar{m}'(z))^2 dz \quad \text{for } s \in \Gamma \end{aligned} \quad (8.11)$$

Let

$$\mu_{0,0}(\xi) = 2 \int_{\Gamma} V_0^{(0)}(\eta) G(\xi, \eta) dS_\eta + c_0(t) \quad \xi \in \Omega \quad (8.12)$$

since  $\mu_{0,0}(\xi) - \mu_0(\xi) \simeq \lambda$ , we first choose  $V_0^{(0)}$  imposing for  $s \in \Gamma$

$$\int_{\mathbb{R}} \mu_{0,0}(0, s) \left[ 1 - \frac{f'(\bar{m}(z))}{f'(1)} \right] \bar{m}'(z) dz = K(s) \int_{\mathbb{R}} (\bar{m}'(z))^2 dz . \quad (8.13)$$

We obtain, since

$$\int_{\mathbb{R}} f'(\bar{m}(z)) \bar{m}'(z) dz = f(1) - f(-1) = 0 \quad (8.14)$$

$$2\mu_{0,0}(\xi) = K(\xi) \int_{\mathbb{R}} (\bar{m}'(z))^2 dz \quad \xi \in \Gamma. \quad (8.15)$$

Inserting (8.12) in (8.15) and integrating over  $\Gamma$  we obtain that

$$4 \int_{\Gamma} dS_\xi \int_{\Gamma} V_0(\eta) G(\xi, \eta) dS_\eta + 2c_0(t) |\Gamma| = 2\pi \int_{\mathbb{R}} (\bar{m}'(z))^2 dz . \quad (8.16)$$

Then from (8.16) we obtain

$$c_0(t) = \frac{1}{2|\Gamma|} \left[ 2\pi \int_{\mathbb{R}} (\bar{m}'(z))^2 dz - 4 \int_{\Gamma} dS_\xi \int_{\Gamma} V_0^{(0)}(\eta) G(\xi, \eta) dS_\eta \right] . \quad (8.17)$$

In this way the constant  $c_0(t)$  of the single layer potential, see (8.12), is written in term of the velocity field  $V_0^{(0)}$  still to be determined. We then insert  $c_0(t)$  as in (8.17) into (8.15) obtaining the equation determining  $V_0^{(0)}$ . We have

$$\mathcal{S}_\Gamma V_0^{(0)}(\eta, \Gamma) = S \left[ K(\eta) - \frac{2\pi}{|\Gamma|} \right] \quad \eta \in \Gamma ,$$

where  $\mathcal{S}_\Gamma$  is the linear operator defined in (10.9). Applying the Dirichlet-Neumann operator, see (10.10), we obtain

$$V_0^{(0)}(\xi, \Gamma) = S \mathcal{T}_\Gamma \left[ K(\cdot) - \frac{2\pi}{|\Gamma|} \right] (\xi) \quad \xi \in \Gamma . \quad (8.18)$$

This determines  $V_0^{(0)}$  and then  $c_0(t)$ , see (8.17). Now that  $V_0^{(0)}$  and  $c_0(t)$  are chosen, we simply choose  $\alpha_1(s)$  so that (8.11) is satisfied. Then for any  $s \in \Gamma$ , Lemma 8.1 assures the existence of the unique solution of (8.8) with  $h_1(0, s) = 0$ , exponentially decaying to zero as  $|z| \rightarrow \infty$ . Since the left hand side of (8.8) is even, the solution  $h_1(\cdot, s)$  is even as function of  $z$ .  $\square$

Note that  $\mu_{0,0}(\cdot, \Gamma_t^{(N)})$ , defined in (8.12) and (8.17) with  $V_0^{(0)}$  given in (8.18) solves for any  $t \in (0, T)$  the (1.8) and (1.9). It is the same already derived in Section 3, see (3.15). The function  $h_1$  determined in Lemma 8.2 is  $\lambda$  different from the one of Section 3. In fact the equation determining  $h_1$  here, see (8.6), has terms of order  $\lambda$  not taken in account in Section 3 when solving for  $h_1$ .

### Second order term in $\lambda$ :

We proceed as before. Since (7.13) and  $\Delta\phi_1 = 0$  in  $\Omega \setminus \mathcal{N}(\lambda_0)$  we obtain

$$\mu_1(\xi) = f'(1)\phi_2(\xi) + \frac{1}{2}f''(1)\phi_1^2(\xi) \quad \xi \in \Omega \setminus \mathcal{N}(\lambda_0). \quad (8.19)$$

which gives, for  $\xi \in \Omega \setminus \mathcal{N}(\lambda_0)$ ,

$$\phi_2(\xi) = \frac{1}{f'(1)} \left[ \mu_1(\xi) - \frac{1}{2}f''(1)\phi_1^2(\xi) \right] \quad (8.20)$$

As done before, we extend the validity of (8.20) globally in  $\Omega$ . We insert (8.20) into (7.12). We add, subtracting to the next order,  $\lambda\alpha_2(s)\bar{m}'(z)$  obtaining

$$\mu_1(\lambda z, s) \left[ 1 - \frac{f'(\bar{m}(z))}{f'(1)} \right] - A_2(z, s) + \lambda\alpha_2(s)\bar{m}'(z) = (\mathcal{L}h_2)(z, s) \quad \xi \in \mathcal{N}\left(\frac{\lambda_0}{2}\right) \quad (8.21)$$

where we set

$$\begin{aligned} A_2(z, s) &= \lambda\Delta\phi_1(\lambda z, s) - f_2(\bar{m}, m_1)(\lambda z, s) + \frac{f'(\bar{m}(z))}{f'(1)} f_2(1, \phi_1)(\lambda z, s) \\ &\quad - K^2(s)z\bar{m}'(z) - K(s)h_1'(z, s) + \alpha_1(s)\bar{m}'(z) \end{aligned} \quad (8.22)$$

All the quantities in (8.22) have been already determined. Further

$$\lim_{|z| \rightarrow \infty} A_2(z, s) = 0 \quad s \in \Gamma \quad (8.23)$$

exponentially fast. Namely, since the exponential convergence of  $\bar{m}(\cdot)$  to  $\pm 1$  and the one of  $\bar{m}'(\cdot)$  and  $h_1'(\cdot, s)$ , for  $s \in \Gamma$ , to 0 we need only to verify that

$$\lim_{|z| \rightarrow \infty} \left[ \lambda\Delta\phi_1(\lambda z, s) - f_2(\bar{m}, m_1)(\lambda z, s) + \frac{f'(\bar{m}(z))}{f'(1)} f_2(1, \phi_1) \right] = 0. \quad (8.24)$$

From (8.7) and (3.14) we have that  $\lambda\Delta\phi_1(\lambda z, s) = \frac{1}{f'(1)}V_0^{(0)}(s)\bar{m}'(z)$ . Then as  $|z| \rightarrow \infty$ , it converges exponentially fast to 0. Further since  $\lim_{|z| \rightarrow \infty} f'(\bar{m}(z)) = f'(1) = f'(-1)$  and  $\lim_{|z| \rightarrow \infty} h_1(z, s) = 0$  exponentially fast, we have (8.24) and therefore (8.23). As done before, we extend (8.21) in  $\mathbb{R}$ .

**Lemma 8.3** *Set*

$$V_1^{(0)}(\xi, \Gamma) = \mathcal{T}_\Gamma \left[ \frac{1}{4}B_1(\cdot) - \frac{1}{4|\Gamma|} \int_\Gamma B_1(s)ds \right] (\xi) \quad \xi \in \Gamma \quad (8.25)$$

where  $B_1(s)$  is defined in (8.28) and  $\mathcal{T}_\Gamma$  is the operator defined in (10.8). Then there are uniquely determined  $\alpha_2(\cdot, \Gamma) \in C^\infty(\Gamma)$  and  $h_2(\cdot, s) \in \Lambda^\infty(\mathbb{R})$ ,  $h_2(0, s) = 0$  with  $s \in \Gamma$ , solution of (8.21). Moreover  $h_2(\cdot, s)$  and its derivatives with respect to  $z$  decay exponentially to 0, as  $z$  tends to  $\pm\infty$ .

*Proof* The solvability condition, see (8.2), is satisfied provided for  $s \in \Gamma$  and  $t \in [0, T]$

$$\int_{\mathbb{R}} \mu_1(\lambda z, s) \left[ 1 - \frac{f'(\bar{m}(z))}{f'(1)} \right] \bar{m}'(z) dz = \int_{\mathbb{R}} A_2(z, s) \bar{m}'(z) dz - \lambda \alpha_2(s) \int_{\mathbb{R}} (\bar{m}'(z))^2 dz \quad (8.26)$$

where  $\mu_1$  is defined in (6.25). The term  $\tilde{\mu}_1$  of  $\mu_1$  has been already completely determined. Still to be determined are, as in the previous case, the constant  $c_1(t)$ , the velocity  $V_1^{(0)}$  and  $\alpha_2(s)$ . Write (8.26) as

$$\int_{\mathbb{R}} \mu_{1,0}(\lambda z, s) \left[ 1 - \frac{f'(\bar{m}(z))}{f'(1)} \right] \bar{m}'(z) dz = B_1(s) - \lambda \alpha_2(s) \int_{\mathbb{R}} (\bar{m}'(z))^2 dz \quad (8.27)$$

where

$$B_1(s) = \int_{\mathbb{R}} \left\{ A_2(z, s) - \tilde{\mu}_1(\lambda z, s) \left[ 1 - \frac{f'(\bar{m}(z))}{f'(1)} \right] \right\} \bar{m}'(z) dz. \quad (8.28)$$

Let

$$\mu_{1,0,0}(\xi) = 2 \int_{\Gamma} V_1^{(0)}(\eta) G(\xi, \eta) dS_\eta + c_1(t) \xi \in \Omega \quad (8.29)$$

since  $\mu_{1,0,0}(\xi) - \mu_{1,0}(\xi) \simeq \lambda$ , we first choose  $V_1$  imposing

$$\int_{\mathbb{R}} \mu_{1,0,0}(0, s) \left[ 1 - \frac{f'(\bar{m}(z))}{f'(1)} \right] \bar{m}'(z) dz = B_1(s) \quad (8.30)$$

We obtain

$$\mu_{1,0,0}(0, s) = \frac{1}{2} B_1(s) \quad s \in \Gamma \quad (8.31)$$

Inserting (8.29) in (8.31) and integrating over  $\Gamma$  we obtain

$$c_1(t) = \frac{1}{|\Gamma_t|} \left[ \frac{1}{2} \int_{\Gamma_t} B_1(\eta) dS_\eta - 2 \int_{\Gamma_t} dS_\xi \int_{\Gamma_t} V_1^{(0)}(\eta) G(\xi, \eta) dS_\eta \right] \quad (8.32)$$

Since  $\int_{\Gamma} V_1^{(0)}(s) ds = 0$ , let  $\mathcal{S}_\Gamma$  be the linear operator defined in (10.9). Then (8.31) can be written as

$$\mathcal{S}_\Gamma V_1^{(0)}(\xi) = \frac{1}{4} B_1(\xi) - \frac{1}{4|\Gamma|} \int_{\Gamma} B_1(s) ds \quad \xi \in \Gamma$$

and applying the Dirichlet-Neumann operator, see (10.10) we obtain (8.25). This determines the (constant in  $\xi$ )  $c_1(t)$ . Now that  $V_1^{(0)}$  and  $c_1(t)$  are determined, we choose  $\alpha_2(s)$  so that (8.27) is satisfied.  $\square$

Notice that  $\mu_{1,0,0}$  solves

$$\begin{aligned} \Delta \mu_{1,0,0} &= 0 \text{ for } \xi \in \Omega \setminus \Gamma \\ \mu_{1,0,0}(s) &= B_1(s) \text{ on } \Gamma. \end{aligned} \quad (8.33)$$

$n$ -th order term in  $\lambda$ ,  $3 \leq n \leq N$ .

As previously, we determine the function  $\phi_n$  for  $\xi \in \Omega \setminus \mathcal{N}(\lambda_0)$  from (7.15). Then, we extend the validity in  $\Omega$  obtaining

$$\phi_n(\xi) = \frac{1}{f'(1)} [\mu_{n-1}(\xi) + \lambda \Delta \phi_{n-1}(\xi) - f_n(\pm 1, \phi_1, \phi_2, \dots, \phi_{n-1})(\xi)] \quad \xi \in \Omega \quad (8.34)$$

We then insert (8.34) into (7.14). We add and subtract (at the next order) the quantity  $\lambda \alpha_n(s) \bar{m}'(z)$ , to the left hand side of (7.14) and we obtain

$$\mu_{n-1}(\lambda z, s) \left[ 1 - \frac{f'(\bar{m}(z))}{f'(1)} \right] - A_n(z, s) + \lambda \alpha_n(s) \bar{m}'(z) = (\mathcal{L}h_n)(z, s) \quad (8.35)$$

where we set

$$\begin{aligned} A_n(z, s) = & -a_{n-1}(z, s) \bar{m}' - \sum_{i=1}^{n-1} [a_{n-i}(z, s) h'_i(z, s)] - \sum_{i=1}^{n-2} b_{n-i}(z, s) \frac{d^2}{ds^2} h_i(z, s) \\ & - \mathbb{1}_{n \geq 4} \sum_{i=1}^{n-3} \left[ c_{n-i}(z, s) \frac{d}{ds} h_i(z, s) \right] - \lambda \Delta \phi_{n-1}(\lambda z, s) \left[ 1 - \frac{f'(\bar{m}(z))}{f'(1)} \right] \\ & + \frac{f'(\bar{m}(z))}{f'(1)} f_n(\pm 1, \phi_1, \phi_2, \dots, \phi_{n-1})(\lambda z, s) - f_n(m_0, m_1, m_2, \dots, m_{n-1})(\lambda z, s) \end{aligned} \quad (8.36)$$

It is easy to verify that for all  $s \in \Gamma$

$$\lim_{|z| \rightarrow \infty} A_n(z, s) = 0$$

exponentially fast. Namely there is no problem for those terms involving  $\bar{m}'$ ,  $h_i(\cdot, s)$  and their derivatives, because of the exponential convergence to zero of all these terms, for all  $s \in \Gamma$ . For the remaining terms recall that  $\lim_{|z| \rightarrow \infty} f'(\bar{m}(z)) = f'(\pm 1)$ ,  $m_i = h_i + \phi_i$  with  $h_i(z, s) \rightarrow 0$ , as  $|z| \rightarrow \infty$  for all  $s \in \Gamma$ , all limits being exponentially fast. Then one obtains immediately

$$\lim_{|z| \rightarrow \infty} \left[ \frac{f'(\bar{m}(z))}{f'(1)} f_n(\pm 1, \phi_1, \phi_2, \dots, \phi_{n-1})(\lambda z, s) - f_n(m_0, m_1, m_2, \dots, m_{n-1})(\lambda z, s) \right] = 0$$

exponentially fast. We extend (8.35) to hold on all of  $\mathbb{R}$ , and regard it as an equation for  $h_n(\cdot, s)$  for  $s \in \Gamma$ .

**Lemma 8.4** *For any positive integer  $n$ ,  $n \leq N$ , set*

$$V_{n-1}^{(0)}(\xi, \Gamma) = \mathcal{T}_\Gamma \frac{1}{4} \left[ B_{n-1}(\cdot) - \frac{1}{|\Gamma|} \int_\Gamma B_{n-1}(s) \right] \quad \text{for} \quad \xi \in \Gamma \quad (8.37)$$

where  $B_{n-1}(s)$  is defined in (8.40). Then there are uniquely determined  $\alpha_n(\cdot, \Gamma) \in C^\infty(\Gamma)$  and  $h_n(\cdot, s) \in L^\infty(\mathbb{R})$  for  $s \in \Gamma$ , with  $h_n(0, s) = 0$  solutions of (8.35). Moreover  $h_n(\cdot, s)$ , for all  $s \in \Gamma$ , and its derivatives with respect to  $z$  decay exponentially to 0 as  $z \rightarrow \pm\infty$ .

**Proof:** The solvability condition is satisfied provided

$$\int_{\mathbb{R}} \mu_{n-1}(\lambda z, s) \left[ 1 - \frac{f'(\bar{m}(z))}{f'(1)} \right] \bar{m}'(z) dz = \int_{\mathbb{R}} A_n(z, s) \bar{m}'(z) dz - \lambda \alpha_n(s) \int_{\mathbb{R}} (\bar{m}'(z))^2 dz. \quad (8.38)$$

Since, see (6.27),  $\mu_{n-1} = \mu_{n-1,0} + \tilde{\mu}_{n-1}$  with  $\tilde{\mu}_{n-1}$  already determined, to satisfy (8.38) we require that

$$\int_{\mathbb{R}} \mu_{n-1,0}(\lambda z, s) \left[ 1 - \frac{f'(\tilde{m}(z))}{f'(1)} \right] \tilde{m}'(z) dz = B_{n-1}(s) - \lambda \alpha_n(s) \int_{\mathbb{R}} (\tilde{m}'(z))^2 dz \quad (8.39)$$

where

$$B_{n-1}(s) = \int_{\mathbb{R}} \left\{ A_n(z, s) - \tilde{\mu}_{n-1}(\lambda z, s) \left[ 1 - \frac{f'(\tilde{m}(z))}{f'(1)} \right] \right\} \tilde{m}'(z) dz \quad (8.40)$$

We set

$$\mu_{n-1,0,0}(\xi) = 2 \int_{\Gamma} V_{n-1}(\eta) G(\xi, \eta) dS_{\eta} + c_{n-1}(t) \quad (8.41)$$

Since  $\mu_{n-1,0}(\xi, t) - \mu_{n-1,0,0}(\xi, t) \simeq \lambda$  we determine  $V_{n-1}$  imposing

$$\int_{\mathbb{R}} \mu_{n-1,0,0}(0, s) \left[ 1 - \frac{f'(\tilde{m}(z))}{f'(1)} \right] \tilde{m}'(z) dz = B_{n-1}(s) \quad (8.42)$$

obtaining

$$\mu_{n-1,0,0}(0, s) = \frac{1}{2} B_{n-1}(s) \quad s \in \Gamma \quad (8.43)$$

Inserting (8.41) in (8.43) and integrating over  $\Gamma$  we obtain

$$c_{n-1}(t) = \frac{1}{|\Gamma|} \left[ \frac{1}{2} \int_{\Gamma} B_{n-1}(\eta) dS_{\eta} - 2 \int_{\Gamma} dS_{\xi} \int_{\Gamma} V_{n-1}^{(0)}(\eta) G(\xi, \eta) dS_{\eta} \right] \quad (8.44)$$

We insert (8.44) into (8.41) obtaining from (8.43)

$$\mathcal{S}_{\Gamma} V_{n-1}^{(0)}(\xi) = \frac{1}{4} \left[ B_{n-1}(\xi) - \frac{1}{|\Gamma|} \int_{\Gamma} B_{n-1}(s) ds \right] \quad \xi \in \Gamma$$

and then (8.37). This determines  $V_{n-1}^{(0)}$  and then  $c_{n-1}(t)$ . We then chose  $\alpha_n$  to satisfy (8.39).  $\square$

### Proof of Theorem 5.2:

To complete the proof of Theorem 5.2 we need to estimate the remainder term, given, see (7.16), by

$$\lambda R_2(\xi, t, \lambda) = \lambda \tilde{R}_2(\xi, t, \lambda) - \lambda^{N+1} \alpha_N(s(\xi), t) \tilde{m}'\left(\frac{d(\xi, \Gamma)}{\lambda}\right) \quad (8.45)$$

Since (7.17) we obtain that

$$\sup_{\xi \in \Omega} \sup_{t \in [0, T]} |R_2(\xi, t, \lambda)| \leq C \lambda^N \quad (8.46)$$

Theorem 5.2 is then proved.  $\square$

## 9. Proof of Theorem 5.3

Set

$$\tilde{m}^{(N)}(\xi, t) = m^{(N)}(\xi, t) - \int_0^t \bar{R}_1(\tau, \lambda) d\tau \quad (9.1)$$

where

$$\bar{R}_1(t, \lambda) = \frac{1}{|\Omega|} \int_{\Omega} R_1(\xi, t, \lambda) d\xi$$

and  $R_1(\xi, t, \lambda)$  is the remainder in Theorem 5.1, defined in (6.9) and estimated in (6.10) and (6.11). Denote

$$\tilde{\mu}^{(N-1)}(\xi, t) = \mu^{(N-1)}(\xi, t) + v(\xi, t) \quad (9.2)$$

where  $v(\xi, t)$  solves

$$\begin{aligned} \Delta v(\xi, t) &= R_1(\xi, t, \lambda) - \bar{R}_1(t, \lambda) \quad \text{for } \xi \in \Omega \\ \frac{\partial}{\partial \nu} v &= 0 \quad \text{on } \partial\Omega \end{aligned} \quad (9.3)$$

with the further requirement

$$\int_{\Omega} v(\xi, t) d\xi = 0 \quad t \in [0, T].$$

Since  $|R(\xi, t, \lambda)| \leq C(T)\lambda^{N-1}$  we have that  $|v(\xi, t)| \leq C\lambda^{N-1}$ . The function  $\tilde{m}^{(N)}$  and  $\tilde{\mu}^{(N-1)}$  satisfy (5.11). Namely the first equation of (5.11) is satisfied by Theorem 5.1 and by construction, see (9.1) and (9.3). The second equation is obtained from Theorem 5.2 adding and subtracting terms to obtain  $\tilde{\mu}^{(N-1)}$  and  $\tilde{m}^{(N)}$ . We obtain

$$\tilde{\mu}^{(N-1)} = \mu^{(N-1)} + v = -\lambda\Delta\tilde{m}^{(N)} + \frac{1}{\lambda}f(\tilde{m}^{(N)}) + R$$

where

$$R \equiv R(\xi, t, \lambda) = \frac{1}{\lambda} \left[ f \left( \tilde{m}^{(N)} + \int_0^t \bar{R}_1(\tau, \lambda) d\tau \right) - f(\tilde{m}^{(N)}) \right] + R_2 + v$$

and  $R_2$  is the remainder in Theorem 5.2, see (8.45). Since  $\bar{R}_1 = \mathcal{O}(\lambda^N)$ ,  $R_2 = \mathcal{O}(\lambda^N)$ ,  $v = \mathcal{O}(\lambda^{N-1})$  and

$$\frac{1}{\lambda} \left[ f \left( \tilde{m}^{(N)} + \int_0^t \bar{R}_1(\tau, \lambda) d\tau \right) - f(\tilde{m}^{(N)}) \right] \leq \frac{C}{\lambda} \int_0^t \bar{R}_1(\tau, \lambda) d\tau = \mathcal{O}(\lambda^{N-1}) \quad (9.4)$$

the second equation of (5.11) is satisfied as well. The (5.13) and (5.14) are satisfied by construction of the  $m^{(N)}$ . Theorem 5.3 is then proved.  $\square$

## Appendix A

### A.1: The Dirichlet–Neumann operator

Let  $G(\xi, \eta)$  be the Green function in  $\Omega$ , with Neumann boundary condition on  $\partial\Omega$ , satisfying the equation

$$\Delta G(\xi, \eta) = \delta(\xi - \eta) - \frac{1}{|\Omega|}, \quad (10.1)$$

so that

$$\int_{\Omega} G(\xi, \eta) d\eta = \int_{\Omega} G(\xi, \eta) d\xi = 0. \quad (10.2)$$

Under the compatibility condition  $\int_{\Omega} f(\xi) d\xi = 0$ , the unique solution of the equation

$$\Delta v(\xi) = f(\xi) \quad \text{for } \xi \in \Omega \quad (10.3)$$

with Neumann boundary conditions in  $\partial\Omega$ , and with  $\int_{\Omega} v(\xi)d\xi = 0$ , is given by

$$v(\xi) = \int_{\Omega} G(\xi, \eta)f(\eta)d\eta . \quad (10.4)$$

All other solutions with Neumann boundary conditions differ from this one by a constant. We will be particularly concerned with certain single layer potentials in what follows. Given a smooth function  $h$  defined on  $\Gamma$ , consider the *single layer potential*

$$\phi_h(\xi) = \int_{\Gamma} G(\xi, \eta)h(\eta)dS_{\eta} ,$$

where  $dS_{\eta}$  denotes the arclength measure along  $\Gamma$ ; this notation is standard in potential theory. The function  $\phi_h$  satisfies Neumann boundary conditions on  $\partial\Omega$ , and satisfies the equation

$$\Delta\phi_h(\xi) = h(\xi)dS_{\xi} - \frac{1}{\Omega} \int_{\Gamma} h(\eta)dS_{\eta} .$$

Clearly there is a discontinuity in the the normal derivatives of  $\phi_h$  across  $\Gamma$ , and we have that

$$h(\xi) = \left[ \frac{\partial}{\partial n} \phi_h \right]_{\Gamma} (\xi) \quad (10.5)$$

where the right hand side is the difference in the normal derivatives at  $\xi \in \Gamma$ :

$$\left[ \frac{\partial}{\partial n} \phi_h \right]_{\Gamma} (\xi) = \left( \frac{\partial \phi_h}{\partial n} \right)_{\Omega_{\Gamma}^+} (\xi) - \left( \frac{\partial \phi_h}{\partial n} \right)_{\Omega_{\Gamma}^-} (\xi) .$$

This is a well known result from potential theory [9]. For  $\xi$  away from  $\Gamma$ ,

$$\Delta\phi_h(\xi) = \frac{1}{\Omega} \int_{\Gamma} h(\eta)dS_{\eta} .$$

Thus the single layer potential is harmonic away from  $\Gamma$  if and only if  $\int_{\Gamma} h(\xi)dS_{\xi} = 0$ . Otherwise, it is subharmonic or superharmonic, according to whether  $\int_{\Gamma} h(\xi)dS_{\xi}$  is positive or negative. Every continuous function  $\phi$  that satisfies the Neumann boundary condition, and is harmonic away from  $\Gamma$ , and which satisfies

$$\int_{\Omega} \phi(\xi)d\xi = 0 \quad (10.6)$$

is the single layer potential of a uniquely determined function  $h$  defined on  $\Gamma$  satisfying

$$\int_{\Gamma} h(\xi)dS_{\xi} = 0 . \quad (10.7)$$

Indeed, if  $\phi_h$  is such a single layer potential, then from (10.2),  $\int_{\Omega} \phi_h(\xi)d\xi = 0$ .

On the other hand, let  $\phi$  be any continuous function that is harmonic on  $\Omega_{\Gamma}^-$  and  $\Omega_{\Gamma}^+$ , and which satisfies (10.6). Define  $h$  in  $\Gamma$  by

$$h(\xi) = \left[ \frac{\partial}{\partial n} \phi \right]_{\Gamma} (\xi) ;$$

we refer to this as the Neumann data for  $\phi$ . By the divergence theorem,

$$\int_{\Gamma} h(\xi) dS_{\xi} = \int_{\Gamma} \nabla \phi \cdot n dS_{\xi} + \int_{\Gamma} \nabla \phi \cdot (-n) dS_{\xi} = \int_{\Omega_{\Gamma}^{-}} \Delta \phi d\xi + \int_{\Omega_{\Gamma}^{+}} \Delta \phi d\xi = 0 .$$

Hence,  $h$  satisfies (10.7).

Notice that  $\phi - \phi_h$  satisfies Neumann boundary conditions and

$$\left[ \frac{\partial}{\partial n} (\phi - \phi_h) \right]_{\Gamma} (\xi) = 0 .$$

This means that  $\phi - \phi_h$  is a constant. Since the integral is zero, it is zero, and so  $\phi = \phi_h$ .

This proves the one to one correspondence between single layer potentials of functions  $h$  satisfying (10.7), and continuous functions  $\phi$  that are harmonic on  $\Omega_{\Gamma}^{-}$  and  $\Omega_{\Gamma}^{+}$ , and which satisfy (10.6).

Next, given a continuous function  $\phi$  that is harmonic on  $\Omega_{\Gamma}^{-}$  and  $\Omega_{\Gamma}^{+}$ , whether or not (10.6) is satisfied, define the function  $g$  on  $\Gamma$  by  $g = \phi|_{\Gamma}$ . We naturally refer to  $g$  as the Dirichlet data for  $\phi$ .

The Neumann data is  $\left[ \frac{\partial}{\partial n} \phi \right]_{\Gamma}$ . The Dirichlet–Neuman operator  $\mathcal{T}_{\Gamma}$  defined by

$$\mathcal{T}_{\Gamma} g = \left[ \frac{\partial}{\partial n} \phi \right]_{\Gamma} \tag{10.8}$$

where  $\phi$  is the continuous function that is Harmonic in  $\Omega_{\Gamma}^{-}$  and  $\Omega_{\Gamma}^{+}$ , and with  $\phi|_{\Gamma} = g$ .

A simple argument shows that  $\mathcal{T}_{\Gamma}$  is a positive Hermitian operator. Indeed, let  $\psi$  be continuous on  $\Omega$ , and harmonic on  $\Omega_{\Gamma}^{-}$  and  $\Omega_{\Gamma}^{+}$ , and with  $\psi|_{\Gamma} = h$ . Then

$$\begin{aligned} \int_{\Gamma} h \mathcal{T}_{\Gamma} g ds &= \int_{\Gamma} \psi \left[ \frac{\partial}{\partial n} \phi \right]_{\Gamma} ds \\ &= \int_{\Omega_{\Gamma}^{+}} \nabla \cdot (\psi(\nabla \phi)) d\xi + \int_{\Omega_{\Gamma}^{-}} \nabla \cdot (\psi(\nabla \phi)) d\xi \\ &= \int_{\Omega} \nabla \psi \cdot \nabla \phi d\xi . \end{aligned}$$

Taking  $h = 1$ , so that  $\psi = 1$ , we further see that the range of  $\mathcal{T}_{\Gamma}$  is orthogonal to the constants. We let  $\mathcal{T}_{\Gamma}$  denote the Friedrichs extension of  $\mathcal{T}_{\Gamma}$ . It is easy to see, and well known, that the form domain of  $\mathcal{T}_{\Gamma}$  is the Sobolev space  $H^{1/2}(\Gamma)$ , and that the kernel consists exactly of the constants. There is an explicit formula for the inverse of  $\mathcal{T}_{\Gamma}$  restricted to the orthogonal complement of the constants; we denote this by  $\mathcal{S}_{\Gamma}$ . Indeed, let  $v$  be any function on  $\Gamma$  with  $\int_{\Gamma} v(s) ds = 0$ . Since the single layer potential  $\phi_v$  for  $v$  has Neumann data  $v$ , all we need to do is to subtract a constant to make this function orthogonal to the constants on  $\Gamma$ , instead of being orthogonal to the constant on  $\Omega$ . Therefore, the inverse  $\mathcal{S}_{\Gamma}$  is given by

$$\mathcal{S}_{\Gamma} v(\xi) = \int_{\Gamma} G(\xi, \eta) v(\eta) dS_{\eta} - \frac{1}{|\Gamma|} \int_{\Gamma} \int_{\Gamma} G(\xi, \eta) v(\eta) dS_{\eta} dS_{\xi} \quad \xi \in \Gamma . \tag{10.9}$$

It is easily checked that this is self adjoint on the orthogonal complement of the constants. Now let  $h$  be an arbitrary smooth function on  $\Gamma$  satisfying  $\int_{\Gamma} h(s) ds = 0$ . Consider the single layer potential

$$\phi(\xi) = \int_{\Gamma} G(\xi, \eta) h(\eta) dS_{\eta} \quad \xi \in \Omega .$$

In general, the Dirichlet data for  $\phi$  does not integrate to zero on  $\Gamma$ , and hence is not directly related to the Neumann data through the Dirichlet–Neumann operator. However, we can correct for this by subtracting a constant: Form the function

$$\tilde{\phi}(\xi) = \phi(\xi) - \frac{1}{|\Gamma|} \int_{\Gamma} \phi(\eta) dS_{\eta} .$$

Then clearly

$$\begin{aligned} \tilde{\phi}|_{\Gamma} &= \mathcal{S}_{\Gamma} h \\ h &= \mathcal{T}_{\Gamma} \tilde{\phi} . \end{aligned} \tag{10.10}$$

We can now express the vector field  $V$  driving the Mullins–Sekerka flow as

$$V = \mathcal{T}_{\Gamma} \left( K - \frac{1}{|\Gamma|} \int_{\Gamma} K(s) ds \right) . \tag{10.11}$$

We close by establishing notation for the two harmonic extension operators that will arise throughout what follows: The *Neumann harmonic extension operator*  $\mathcal{E}_{\Gamma,N}$  is defined by

$$(\mathcal{E}_{\Gamma,N}v)(\xi) = \int_{\Gamma} G(\xi, \eta) v(\eta) dS_{\eta} - \frac{1}{|\Gamma|} \int_{\Gamma} \int_{\Gamma} G(\xi, \eta) v(\eta) dS_{\eta} dS_{\xi} \quad \xi \in \Omega , \tag{10.12}$$

where  $v$  is a function on  $\Gamma$  satisfying

$$\int_{\Gamma} v(\xi) dS_{\xi} = 0 .$$

Notice that  $(\mathcal{E}_{\Gamma,N}v)(\xi)$  is the unique function that is continuous on  $\Omega$ , harmonic on  $\Omega \setminus \Gamma$  satisfying Neumann boundary conditions on  $\partial\Omega$ , with Neuman data  $v$  on  $\Lambda$ , and with zero integral over  $\Gamma$ .

The *Dirichlet harmonic extension operator*  $\mathcal{E}_{\Gamma,D}$  is defined by setting  $\mathcal{E}_{\Gamma,D}g(\xi)$  to be the harmonic function  $\phi$  on  $\Omega \setminus \Gamma$  with Neumann boundary conditions on  $\partial\Gamma$ , and with  $\phi|_{\Gamma} = g$ . Here, there is no restriction on the integral of  $g$  over  $\Gamma$ .

Naturally, the Dirichlet extension can be expressed in terms of the Neumann extension and the Dirichlet–Neumann operator. We have from (10.9) and (10.12) that

$$\mathcal{E}_{\Gamma,D}g(\xi) = \mathcal{E}_{\Gamma,N} \left( \mathcal{T}_{\Gamma} \left( g - \frac{1}{|\Gamma|} \int_{\Gamma} g(\eta) dS_{\eta} \right) \right) (\xi) + \frac{1}{|\Gamma|} \int_{\Gamma} g(\eta) dS_{\eta} \quad \xi \in \Omega . \tag{10.13}$$

## A.2: The expansion in $\lambda$ of the Laplacian in local coordinates.

Let  $f(z, s)$ ,  $z = \frac{d}{\lambda}$ , be a  $C^2$  function from  $\mathbb{R} \times \Gamma$  to  $\mathbb{R}$ . Then, in dimension  $d = 2$ , we have that

$$\begin{aligned} \lambda^2 \Delta f(z, s) &= \frac{1}{1 - K(s)\lambda z} \left\{ ((1 - K(s)\lambda z) f_z)_z + \lambda^2 \left( \frac{f_s}{1 - K(s)\lambda z} \right)_s \right\} \\ &= f_{zz} - \lambda K(s) f_z \frac{1}{1 - K(s)\lambda z} + \lambda^2 \frac{f_{ss}}{(1 - K(s)\lambda z)^2} + \lambda^3 f_s \frac{\frac{d}{ds} K(s) z}{(1 - K(s)\lambda z)^3} \end{aligned} \tag{10.14}$$

Recalling that, for  $|x| < 1$

$$\frac{1}{(1-x)} = \sum_{n=0}^{\infty} x^n; \quad \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}; \quad \frac{1}{(1-x)^3} = \frac{1}{2} \sum_{n=0}^{\infty} n(n-1) x^{n-2}$$

we rewrite (10.14) as the following

$$\lambda^2 \Delta f = f_{zz} + \sum_{n=0}^{\infty} \lambda^{n+1} \{a_{n+1}(z, s)f_z + b_{n+1}(z, s)f_{ss} + c_{n+1}(z, s)f_s\} \quad (10.15)$$

where

$$\begin{aligned} a_{n+1}(z, s) &= -K^{n+1}(s)z^n, \\ b_{n+1}(z, s) &= nK^{n-1}(s)z^{n-1}, \\ c_{n+1}(z, s) &= \frac{1}{2}n(n-1)z^{n-1}K^{n-2}(s)\frac{d}{ds}K(s). \end{aligned} \quad (10.16)$$

### References

- [1] N. Alikakos, P.W. Bates, X Chen, *Convergence of the Cahn-Hilliard equation to the Hele-Shaw model*. Arch. Rat. Mech. Anal. **128**, 165–205 (1994).
- [2] S. Bastea, R. Esposito, J.L. Lebowitz, R. Marra, *Binary fluids with long range segregating interaction. I: Derivation of kinetic and hydrodynamic equations*. J. Stat. Phys., **101**, 1087–1136 (2000).
- [3] L. Caffarelli, N.E. Muller, *An  $L^\infty$  bound for the Cahn-Hilliard equation*. Arch. Rat. Mech. Anal. **133** 129–144 (1999).
- [4] R. Caffisch, *The fluid dynamical limit of the nonlinear equation*. Comm. Pure and Applied Math. **33** 651–666 (1980).
- [5] C. Cercignani, R. Illner, M. Pulvirenti, *The Mathematical Theory of Dilute Gasses*. Springer-Verlag, Berlin (1994).
- [6] X. Chen, *The Hele-Shaw Problem and Area-Preserving Curve-Shortening Motions*. Arch. Rat. Mech. Anal. **123** 117–151 (1993).
- [7] X. Chen, *Global asymptotic limit of solutions of the Cahn-Hilliard Equation*. J. Diff. Eq. **44**, no. 2, 262-311 (1996)
- [8] X. Chen, J. Hong, F. Yi, *Existence, uniqueness, and regularity of classical solutions to the Mullins-Sekerka problem*. Comm. Partial Diff. Equ. **21**,no 11-12, 1705-1726 (1996)
- [9] E. DiBenedetto, *Partial differential equations*. Boston Birkhauser,(1995).
- [10] J. Escher, G Simonett, *Classical solutions of multidimensional Hele-Shaw models*. SIAM J. Math. Anal. **28**, 1028–1047 (1997).
- [11] G. Giacomin, J. Lebowitz, *Phase segregation dynamics in particle systems with long range interactions I: macroscopic limits* J. Stat. Phys. **87**, 37–61 (1997).
- [12] G. Giacomin, J. Lebowitz, *Phase segregation dynamics in particle systems with long range interactions II: interface motions* SIAM J. Appl. Math. **58**, No 6, 1707–1729 (1998).
- [13] R.L. Pego, *Front migration in the nonlinear Cahn-Hilliard equation*. Proc. R. Soc. Lond.A **422**, 261–278 (1989).
- [14] P. Fife and J.B. McLeod, *The approach of solutions of nonlinear diffusion equations to travelling front solutions*. Arch. Rat. Mech. Anal. **65**, 335–361, (1977)
- [15] B. Stoth, *Convergence of the Cahn-Hilliard equation to the Mullins-Sekerka problems in spherical symmetry*. J. Diff. Eqns. **125**, 154–183 (1996).

- [16] S. Yu, *Hydrodynamic limits with shock waves of the Boltzmann equation* to appear in Comm. Pure and Applied Math., 2004