PROBABILISTIC INVESTIGATIONS ON THE EXPLOSION OF SOLUTIONS OF THE KAC EQUATION WITH INFINITE ENERGY INITIAL DISTRIBUTION

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Abstract

In Gabetta and Regazzini (2006b), the authors show, using a probabilistic approach, that finiteness of the initial energy (second moment) is necessary and sufficient in order that the solution of the Kac analogue of Boltzmann's equation weakly converges (C_b-convergence) to a probability measure on $\mathbb{R}$. While the sufficiency of the above condition is well-known, there seems not to have been any proof of its necessity, even though it is intuitively clear on physical grounds. However, the probabilistic investigation of Gabetta and Regazzini (2006b) provides the first actual proof. In the present paper, the study of the behaviour of the solution-when the initial energy is infinite-is made more specific by proving that such solution vaguely converges (C_0 convergence) to the measure which is identically null on the Borel sets of $\mathbb{R}$. This is tantamount to asserting that the image of the solution in the usual one-point compactification of $\mathbb{R}$ weakly converges to the unit mass at the point at infinity. Estimations of the rate of convergence (explosion of the mass, in this case) are provided both in $\mathbb{R}$ and in the one-point compactification (the latter, in terms of the Prokhorov distance). Finally, an analogous result is derived, as a corollary, for sums of i.i.d. random variables $\tilde{x}_1, \tilde{x}_2, \ldots$, having infinite second moment, $T_n := \sum_{j=1}^{\nu_n} \lambda_{j,n}\tilde{x}_j$, with $\max_{1\leq j\leq \nu_n} |\lambda_{j,n}| \to 0$ (as $n \to +\infty$), and $\sum_{j=1}^{\nu_n} \lambda_{j,n}^2 = 1$ $(n = 1, 2, \ldots)$.

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1 Presentation

The Kac analogue of the Boltzmann equation, under the usual hypothesis that the initial velocity probability distribution μ_0 is absolutely continuous with density function f_0, can be written as

$$\begin{cases} \dfrac{\partial}{\partial t} f(v,t) = \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}\times[0,2\pi)} \{f(v\cos\theta - w\sin\theta, t) f(v\sin\theta + w\cos\theta, t) \\ \qquad\qquad\qquad\qquad\qquad\qquad - f(v,t)f(w,t)\} dw d\theta \\ f(v,0) = f_0(v) \qquad (t>0, v\in\mathbb{R}) \end{cases}$$

where $f(\cdot,t)$ denotes the probability density function of the velocity at time t. In terms of the Fourier transforms φ_0 and $\varphi(\cdot,t)$ of f_0 and $f(\cdot,t)$ respectively, the previous equation becomes

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi(\xi,t) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \varphi(\xi\sin\theta,t)\varphi(\xi\cos\theta,t)d\theta - \varphi(\xi,t) \\ \varphi(\xi,0) = \varphi_0(\xi) \qquad (t>0, \xi\in\mathbb{R}). \end{cases} \tag{1}$$

See Bobylev (1984). In the present paper we deal meanly with (1), since it is valid even when φ_0 is the Fourier-Stieltjes transform (characteristic function) of any, not necessarily absolutely continuous, initial distribution μ_0. Equation (1) has a unique solution that can be expressed by *Wild sums*. See Wild (1951). More precisely,

$$\phi(\xi,t) = \sum_{n\geq 1} e^{-t}(1-e^{-t})^{n-1}\hat{Q}_n^+(\xi;\varphi_0) \qquad (t\geq 0, \xi\in\mathbb{R}) \tag{2}$$

with $\hat{Q}_n^+$ determined recursively according to

$$\hat{Q}_n^+(\xi;\varphi_0) = \frac{1}{n-1}\sum_{j=1}^{n-1}\hat{Q}_{n-j}^+(\xi;\varphi_0)\circ\hat{Q}_j^+(\xi;\varphi_0) \qquad (n=2,3,\dots) \tag{3}$$

with $\hat{Q}_1^+ := \varphi_0$ and $\circ$ defined as it follows. Given any pair of characteristic functions $\hat{g}_1$, $\hat{g}_2$, $\hat{g}_1\circ\hat{g}_2$ is defined by

$$\hat{g}_1\circ\hat{g}_2(\xi) = \frac{1}{2\pi}\int_0^{2\pi}\hat{g}_1(\xi\cos\theta)\hat{g}_2(\xi\sin\theta)d\theta \qquad (\xi\in\mathbb{R}).$$

The main goal of the present paper is the study of the vague convergence-i.e. $C_0(\mathbb{R})$ convergence-of the probability measure $\mu(\cdot, t)$ corresponding to $\varphi(\cdot, t)$ when μ_0 has infinite energy, i.e.

$$\int_{\mathbb{R}} v^2 \mu_0(dv) = +\infty. \tag{4}$$

As a consequence of the following proposition, that will be proved in Section 3, one has that $\mu(\cdot, t)$ vaguely converges to the null measure on $\mathbb{R}$, that is

$$\mu(I, t) \to 0 \qquad (t \to +\infty, I = \text{any bounded interval}), \tag{5}$$

if (4) is valid. This completes Theorem 2 in Gabetta and Regazzini (2006b), by specifying what happens with an infinite initial energy.

Theorem 1 *If the initial probability measure μ_0 in problem (1), or (2), satisfies (4), then, for any $\eta \in (0, 1)$ there is a time t_η such that*

$$\mu([-R, R], t) \leq \frac{1}{m_2(L_t)^{1/2}} \left\{ \frac{\sqrt{8}R}{m_0(L_t) - \eta} + e^{-t(8/(3\pi)-3/4)} \right\}, \qquad 1 > m_0(L_t) > \eta$$

hold true for every $t > t_\eta$, $R \geq 4/(\sqrt{2}(m_0(L_{t_\eta}) - \eta))$ and

$$L_t := \exp\{t(1 - \frac{8}{3\pi})\}$$

with

$$m_k(L) := \int_{[-L,L]} x^k \mu_0(dx) \qquad (k = 0, 1, 2, \ldots).$$

Thus, if the initial energy is infinite, the mass "explodes" as $t \to +\infty$. Even though the occurrence of this phenomenon seems to be intuitively likely on physical grounds, there seems not to have been any proof of it, much less of the rate at which the explosion occurs.

A more vivid picture can be obtained by assuming that each $\mu(\cdot, t)$, $t > 0$, is defined on the straight line $l_1 := \{(x, y) \in \mathbb{R}^2 : y = 1\}$, and by "projecting" each $\mu(\cdot, t)$ onto the circle S_1 of centre $(0, 1/2)$ and radius $1/2$, according to the following familiar procedure.

Firstly define the projection of each point P of l_1 onto S_1 as the point $P' \neq O$ (the origin) in which the straight line OP joining O and P, intersects S_1. It is well-known that S_1 is homeomorphic to the one-point compactification of $\mathbb{R}$ with ∞ (the point at infinity)$= O$. Then, if θ_P denotes the angle formed by the semiaxis of the positive abscissae and the line OP, P must coincide with $(\cot\theta_P, 1)$ and θ_P is also the length of the arc OP' (distance between O and P' on S_1). Denote such a length by $d(O, P)$. Next, the "projection" $\mu'(\cdot, t)$ of $\mu(\cdot, t)$ onto S_1 is defined in such a way that, for any pair of angles $0 < \theta_1 < \theta_2 < \pi$, $\mu'((\theta_1, \theta_2), t)$ satisfies

$$\mu'((\theta_1, \theta_2), t) = \mu((\cot\theta_2, \cot\theta_1), t).$$

Analogously, the limiting (null) measure of $\mu(\cdot, t)$ found in Theorem 1 is sent to the unit mass at O, δ_O. Therefore, distance between $\mu'(\cdot, t)$ and δ_O-in any metrization of the weak convergence of probabilities on S_1, i.e. $C_b(S_1)$-convergence-provides a quantification of the speed of explosion of the mass. In particular, we study the problem with respect to the *Prokhorov metric* ρ. See, for example, Section 11.3 of Dudley (2002). Recall that, for any pair of probability measures P and Q on some metric space (S, d), ρ is defined by

$$\rho(P, Q) := \inf\{\varepsilon > 0 : P(A) \leq Q(A^\varepsilon) + \varepsilon, \text{ for all Borel sets } A\}$$

with $A^\varepsilon := \{y \in S : d(x, y) < \varepsilon, \text{ for some } x \in A\}$.

A propos of the "projection" μ' and δ_O, one has

Theorem 2 *Let ρ stand for the Prokhorov metric for probability measures on the circle S_1 and μ_0 satisfy (4). Then, for every $\eta \in (0, 1)$,*

$$\rho(\mu'(\cdot, t), \delta_O) \leq \frac{1}{m_2(L_t)^{1/4}} \left\{ g_2(t_\eta) + \sqrt{g_2(t_\eta)^2 + g_1(t_\eta)} \right\}, \qquad 1 > m_0(L_t) > \eta$$

are valid for any $t \geq t_\eta$ (the same as in Theorem 1) and

$$L_t := \exp\{t(1 - \frac{8}{3\pi})\},$$

$$g_2(t) := \frac{1}{2m_2(L_t)^{1/4}} e^{-t(8/(3\pi) - 3/4)}, \quad g_1(t) := \frac{\sqrt{8}}{m_0(L_t) - \eta}.$$

The methods, we shall see in Section 3 to prove Theorems 1 and 2, can be applied to the study of the asymptotic behaviour of the probability distribution of the sum

$$T_n := \sum_{j=1}^{\nu_n} \lambda_{j,n} \tilde{x}_j \tag{6}$$

when $\tilde{x}_1$, $\tilde{x}_2, \ldots$, are *independent* and *identically distributed* random variables and, for each n, $(\lambda_{j,n})_{j=1,\ldots,\nu_n}$ are real numbers and ν_n a positive integer such that

$$\lambda_{(n)} := \max_{1 \leq j \leq \nu_n} |\lambda_{j,n}| \to 0 \quad (\text{as } n \to +\infty), \tag{7}$$

and

$$\sum_{j=1}^{\nu_n} \lambda_{j,n}^2 = 1 \qquad (n = 1, 2, \ldots). \tag{8}$$

Solutions of the above problem can be easily found by resorting to the central limit theorem if the variance of each $\tilde{x}_j$ is finite. Instead, when the variance is infinite, there seems to have been only partial and isolated answers like, e.g., an example contained in Crimaldi (2002) in which $\lambda_{j,n} = 1/\sqrt{n}$, $\nu_n = n$ and the common distribution of the $\tilde{x}_j$ meets a few extra-conditions. It is this state of affairs that suggests to make explicit

Corollary 3 *Let the common probability distribution, μ_0, of the independent random variables $\tilde{x}_j$, in (6), satisfy (4), and let $(\lambda_{j,n})_{j=1,\ldots,\nu_n}$ meet (7) and (8). Then, for any $\eta \in (0,1)$ there is N_η such that*

$$Pr\{-R \leq T_n \leq R\} \leq \frac{1}{m_2(L_n)^{1/2}} \left\{ \frac{\sqrt{8}R}{m_0(L_n) - \eta} + \lambda_{(n)} \right\}, \quad 1 > m_0(L_n) > \eta$$

are valid for every $n \geq N_\eta$, $R \geq 4/(\sqrt{2}(m_0(L_{N_\eta}) - \eta))$ with

$$L_n := (\lambda_{(n)})^{-1}.$$

Moreover, if μ'_n denotes the image of the law of T_n, through the aforesaid mapping from the line l_1 to the circle S_1, then

$$\rho(\mu'_n, \delta_O) \leq \frac{1}{m_2(L_n)^{1/4}} \left\{ g_2(N_\eta) + \sqrt{g_2(N_\eta)^2 + g_1(N_\eta)} \right\}$$

holds for any $n \geq N_\eta$.

With this, the presentation of the main results ends. Between them and their proof, in Section 3, we insert a new section which explains the general ideas of the proofs and recalls a few well-known statements to be used in Section 3.

We have obtained similar results for the case of the spatially homogeneous Boltzmann equation for Maxwellian molecules. The analysis is, however, quite different so that there is little methodological overlap with the present paper, which moreover provides more explicit bounds. See Carlen, Gabetta and Regazzini (2006).

2 Preliminaries to proofs

There have been several previous investigations of the Kac equation using probabilistic methods. For example, Tanaka (1973, 1978), Murata and Tanaka (1974) proved a contraction property in the 2-Wasserstein metric. This is only defined on probability measures with finite second moments, however. For other probabilistic investigations see McKean (1963). Here, the infinite energy (variance) setting requires a new approach.

The argument used to prove Theorem 1 rests on two main ideas. The former stems from the remark that the Wild series, as re-written by McKean McKean (1966, 1967), highlights that $\mu(\cdot, t)$ can be viewed as probability distribution of a linear combination of a random number (with probability law depending on t) of identically distributed random variables, with common distribution μ_0 and random coefficients. The latter relates to the fact that μ_0 can be thought of as probability law of

$$\alpha Y + (1-\alpha)Z \tag{9}$$

when α, Y and Z are stochastically independent random variables defined in such a way that: α belongs to $\{0, 1\}$ and takes value 1 with probability $m_0(L) \in (0, 1)$, for a suitable choice of L; Y and Z have distribution (on the Borel class of $\mathbb{R}$)

$$\underline{\mu}_L(\cdot) := \frac{1}{m_0(L)}\mu_0([-L, L] \cap \cdot) \tag{10}$$

and

$$\overline{\mu}_L(\cdot) := \frac{1}{1-m_0(L)}\mu_0([-L, L]^c \cap \cdot) \tag{11}$$

respectively.

For the sake of explanation one starts by recalling the theorem of McKean which states that $\hat{Q}_n^+$ (see (3)), for every $n \geq 2$ can be written as

$$\hat{Q}_n^+(\xi; \varphi_0) = \sum_{\gamma \in G(n)} p_n(\gamma) C_\gamma(\xi; \varphi_0) \qquad (\xi \in \mathbb{R}) \tag{12}$$

where:
$G(n)$ is the set of all trees γ with n leaves in which each node has either 0 or two "children" (then the cardinality of $G(n)$ is given by the Catalan number $C_n = \binom{2n-2}{n-1}/n$);
$p_n(\cdot)$ is a specific probability on $G(n)$;
$C_\gamma(\cdot; \varphi_0)$, called n-fold product of φ_0 with itself, performed according to γ, is

given by

$$C_\gamma(\xi;\varphi_0) = \int_{[0,2\pi)^\infty} \Big(\prod_{l\in\gamma} \varphi_0(\pi_l;\xi)\Big) u^\infty(d\theta) \qquad (\gamma \in G := \cup_n G(n), \xi \in \mathbb{R}), \tag{13}$$

u^∞ being the probability measure on the Borel subsets of $[0,2\pi)^\infty$ which makes the coordinates stochastically independent with the same *uniform* continuous distribution on $[0,2\pi)$ and, for each leaf l of γ,

$$\pi_l = \pi(l;\gamma,\theta) = \tau_1 \cdots \tau_{\delta_l} \qquad (\gamma \in G, \theta = (\theta_1,\theta_2,\ldots)), \tag{14}$$

with

$\tau_{\delta_l} = \cos\theta_{\delta_l}$ if l is a "left child" or $\tau_{\delta_l} = \sin\theta_{\delta_l}$ if l is a "right child",

$\tau_{\delta_{l-1}} = \cos\theta_{\delta_{l-1}}$ or $\tau_{\delta_{l-1}} = \sin\theta_{\delta_{l-1}}$ depending on the parent of l is, in its turn, a "left child" or a "right child", and so forth;

$\delta_l = \delta_l(\gamma)$, called the depth of l, is the number of generations which separate l from the "root" of γ (the node at the top of γ).

For more information on (12)-(14) see, in addition to McKean (1966, 1967), Carlen, Carvalho and Gabetta (2000, 2005), Gabetta and Regazzini (2006a). Here it suffices to recall that

$$\sum_{l=1}^{\tilde{\nu}_t} \pi_l^2 = 1.$$

Next one quotes the argument developed in Gabetta and Regazzini (2006b) to show that (2) represents the characteristic function of a sum of random variables defined on the probability space we are about to define. Set

$$\Omega = (\mathbb{N}\times G)\times[0,2\pi)^\infty \times \{0,1\}^\infty \times (\mathbb{R}^2)^\infty,$$

denote the coordinate maps of Ω by

$$(\tilde{\nu}_t,\tilde{\gamma}), (\tilde{\theta}_1,\tilde{\theta}_2,\ldots), (\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots), ((\tilde{Y}_1,\tilde{Z}_1),(\tilde{Y}_2,\tilde{Z}_2),\ldots),$$

and consider the σ-algebra

$$\mathscr{F} = \mathscr{P}(\mathbb{N}\times G)\otimes\mathscr{B}([0,2\pi)^\infty)\otimes\mathscr{B}(\{0,1\}^\infty)\otimes\mathscr{B}((\mathbb{R}^2)^\infty),$$

where, for any set S, $\mathscr{P}(S)$ stands for the power set of S. At this stage equip $(\Omega,\mathscr{F})$ with the probability measure P_t characterized by assigning value

$$P_t(M\times T\times A\times D\times B) = \sum_{n\in M} q_t(n)p_n(\gamma)u^\infty(A)\beta(D)(\underline{\mu}_L\times\overline{\mu}_L)^\infty(E)$$

to any rectangle with $M \subset \mathbb{N}$, $T \subset G$, $A \in \mathscr{B}([0, 2\pi)^\infty)$, $D \in \mathscr{B}(\{0,1\}^\infty)$, $E \in \mathscr{B}((\mathbb{R}^2)^\infty)$, β being the law of a Bernoulli sequence with probability of success $m_0(L)$ and $q_t(n) = e^{-t}(1 - e^{-t})^{n-1}$ for $n = 1, 2, \ldots$.

In view of the independence induced by P_t among the coordinate maps, one can say that the random variables

$$\tilde{\alpha}_i \tilde{Y}_i + (1 - \tilde{\alpha}_i)\tilde{Z}_i \qquad (i = 1, 2, \ldots)$$

are independent with the same distribution μ_0. It follows that the mixture of products of characteristic functions defined through (2), (12) and (13) is the characteristic function of the random variable

$$\tilde{V}_t := \sum_{l=1}^{\tilde{\nu}_t} \pi_l \{\tilde{\alpha}_l \tilde{Y}_l + (1 - \tilde{\alpha}_l)\tilde{Z}_l\}. \tag{15}$$

The following identities from Gabetta and Regazzini (2006a) are frequently used in the rest of this paper:

$$E_t\Big(\sum_{l=1}^{\tilde{\nu}_t} \Big(\frac{c}{2}\Big)^{\delta_l(\tilde{\gamma})} \mid \tilde{\nu}_t\Big) = \frac{\Gamma(c + \tilde{\nu}_t - 1)}{\Gamma(c)\Gamma(\tilde{\nu}_t)} \tag{16}$$

$$E_t\Big(\sum_{l=1}^{\tilde{\nu}_t} \Big(\frac{c}{2}\Big)^{\delta_l(\tilde{\gamma})}\Big) = e^{-t(1-c)} \tag{17}$$

and

$$E_t\Big(\sum_{l=1}^{\tilde{\nu}_t} |\pi_l|^m \mid \tilde{\nu}_t\Big) = \frac{\Gamma(2\beta_m + \tilde{\nu}_t - 1)}{\Gamma(2\beta_m)\Gamma(\tilde{\nu}_t)} \tag{18}$$

$$E_t\Big(\sum_{l=1}^{\tilde{\nu}_t} |\pi_l|^m\Big) = e^{-t(1-2\beta_m)} \tag{19}$$

with

$$\beta_m = \frac{1}{2\pi}\int_0^{2\pi} |\sin\theta|^m \, d\theta \qquad (m = 0, 1, 2, \ldots).$$

Finally, the following statement is worth recalling: *If (4) holds, then for every $\varepsilon \in (0,1)$ there is L_ε such that*

$$\frac{m_2(L)}{m_0(L)} - \left(\frac{m_2(L)}{m_0(L)}\right)^2 \geq (1 - \varepsilon) m_2(L) \tag{20}$$

holds true for every $L > L_\varepsilon$.

A proof can be found in the Khintchine (1935) paper on the characterization of the domain of attraction of a Gaussian law. See also Theorem 1 in Section 35 of Gnedenko Kolmogorov (1954).

3 Proofs

Set

$$\tilde{S}_{1,t} := \sum_{l=1}^{\tilde{\nu}_t} \pi_l \tilde{\alpha}_l \tilde{Y}_l, \qquad \tilde{S}_{2,t} := \sum_{l=1}^{\tilde{\nu}_t} \pi_l (1-\tilde{\alpha}_l) \tilde{Z}_l,$$

$$\tilde{s}_t := (\tilde{\nu}_t, \tilde{\gamma}, \tilde{\theta}, \tilde{\alpha}), \qquad \tilde{\theta} := (\tilde{\theta}_1, \tilde{\theta}_2, \ldots), \quad \tilde{\alpha} := (\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots)$$

$$E(\tilde{S}_{1,t} \mid \tilde{s}_t) = \sum_{l=1}^{\tilde{\nu}_t} \pi_l \tilde{\alpha}_l M_1(L) =: M_1(L) a(\tilde{s}_t)$$

$$Var(\tilde{S}_{1,t} \mid \tilde{s}_t) = \sum_{l=1}^{\tilde{\nu}_t} \pi_l^2 \tilde{\alpha}_l^2 \overline{M_2}(L) =: \overline{M_2}(L) b(\tilde{s}_t)^2$$

and, for $k = 1, 2, \ldots,$

$$M_k(L) := \frac{m_k(L)}{m_0(L)}, \qquad \overline{M_k}(L) := \int_{\mathbb{R}} |x - M_1(L)|^k \underline{\mu}_L(dx).$$

Moreover, let $F_{1,t}$ and $F_{2,t}$ be the conditional probability distribution functions of $\tilde{S}_{1,t}$ and $\tilde{S}_{2,t}$, given $\tilde{s}_t$, and $F^*_{1,t}$ be the conditional probability distribution function of

$$\tilde{S}_{1,t} := \frac{\tilde{S}_{1,t} - M_1(L) a(\tilde{s}_t)}{\overline{M_2}(L)^{1/2} b(\tilde{s}_t)}$$

given $\tilde{s}_t$.

To prove Theorem 1, use conditional independence and (15) to write

$$\begin{aligned} P_t\{-R \le \tilde{V}_t \le R\} &= E_t[\int_{\mathbb{R}} \{F_{1,t}(R-x) - F_{1,t}(-R-x)\} \, dF_{2,t}(x)] \\ &= E_t[\int_{\mathbb{R}} \{F^*_{1,t}((R-x)^*) - F^*_{1,t}((-R-x)^*)\} \, dF_{2,t}(x)] \\ &\le E_t(I_1 + I_2 + I_3) + P_t(C^c_\eta) \end{aligned}$$

where

$$C_\eta := \{|b(\tilde{s}_t)^2 - m_0(L)| \le \eta\} \qquad (\eta \in (0, m_0(L))),$$

$$I_1 := \mathbb{I}_{C_\eta} \int_{\mathbb{R}} |F^*_{1,t}((R-x)^*) - \Phi((R-x)^*)| \, dF_{2,t}(x)$$

$$I_2 := \mathbb{I}_{C_\eta} \int_{\mathbb{R}} |F^*_{1,t}((-R-x)^*) - \Phi((-R-x)^*)| \, dF_{2,t}(x)$$

$$I_3 := \mathbb{I}_{C_\eta} \int_{\mathbb{R}} \{\Phi((R-x)^*) - \Phi((-R-x)^*)\} \, dF_{2,t}(x)$$

Φ being the standard Gaussian distribution function.

Now, from the Berry-Esseen inequality (see Theorem 3 with $\delta = 1$ in Section 9.1 of Chow and Teicher (1997)) one gets

$$\begin{aligned} E_t(I_1+I_2) &\le 2c_1 E_t\left(\frac{\sum_{k=1}^{\tilde\nu_t}|\pi_k|^3\tilde\alpha_k^3\overline{M_3}(L)}{\overline{M_2}(L)^{3/2}b(\tilde s_t)^3}\mathbb{I}_{C_\eta}\right) \\ &\le 2c_1\frac{\overline{M_3}(L)m_0(L)}{\overline{M_2}(L)^{3/2}(m_0(L)-\eta)^{3/2}}E_t\big(\sum_{k=1}^{\tilde\nu_t}|\pi_k|^3\big) \\ &= 2c_1\frac{\overline{M_3}(L)m_0(L)}{\overline{M_2}(L)^{3/2}(m_0(L)-\eta)^{3/2}}e^{-t(1-2\beta_3)} \end{aligned} \tag{21}$$

(from (19)). On the other hand, from the symmetry of Φ',

$$E_t(I_3) \le \frac{2R}{\sqrt{2\pi}(m_0(L)-\eta)^{1/2}\overline{M_2}(L)^{1/2}}. \tag{22}$$

Finally, from the Bienaymé-Chebyshev inequality,

$$P_t(C_\eta^c) \le \frac{1}{\eta^2}m_0(L)(1-m_0(L))E_t\big(\sum_{k=1}^{\tilde\nu_t}\tilde\pi_k^4\big) = \frac{1}{\eta^2}m_0(L)(1-m_0(L))e^{-t/4}. \tag{23}$$

Thus, (21)-(23) yield

$$\begin{aligned} P_t\{-R\le\tilde V_t\le R\} &\le \frac{4c_1L}{\overline{M_2}(L)^{1/2}(m_0(L)-\eta)^{3/2}}e^{-t(1-2\beta_3)} \\ &\quad + \frac{2R}{\overline{M_2}(L)^{1/2}\sqrt{2\pi}(m_0(L)-\eta)^{1/2}} + \frac{1}{\eta^2}m_0(L)(1-m_0(L))e^{-t/4} \end{aligned} \tag{24}$$

and, in view of (20) with $\varepsilon = \eta$ and the fact that $c_1 < 1$ (see Shiganov (1989)), a new bound can be given for the right hand side of (24), i.e.

$$\frac{1}{m_2(L)^{1/2}}\left\{\frac{\sqrt{2}R}{m_0(L)-\eta} + \frac{4L}{(m_0(L)-\eta)^2}e^{-t(1-2\beta_3)} + \frac{m_2(L)^{1/2}(1-m_0(L))}{\eta^2}e^{-t/4}\right\}$$

valid for every $L > L_\eta$.

Now, take $L = L_t := \exp\{-t(1-2\beta_3)\}$ and t_η such that $L_t > L_\eta$, $m_0(L_t) > 13/20$ and $(1-m_0(L_t)) \le \eta^2$ whenever $t \ge t_\eta$, along with

$$R \ge \frac{4}{\sqrt{2}(m_0(L_{t_\eta})-\eta)}$$

to obtain the bound

$$\frac{1}{m_2(L)^{1/2}} \left\{ \frac{\sqrt{8}R}{m_0(L) - \eta} + e^{-t(\frac{8}{3\pi} - \frac{3}{4})} \right\}.$$

This completes the proof of Theorem 1.

To prove Theorem 2, define X_t to be a random element with distribution $\mu'(\cdot, t)$ on the circle S_1, and take $\varepsilon \in (0, \pi/2)$. Then

$$\begin{aligned} \mu'(d(O, X_t) > \varepsilon) &= \mu(-\cot\varepsilon, \cot\varepsilon), t) \\ &\leq \frac{1}{m_2(L_t)^{1/2}} \left\{ \frac{\sqrt{8}\cot\varepsilon}{m_0(L_t) - \eta} + e^{-t(\frac{8}{3\pi} - \frac{3}{4})} \right\} \\ &\leq \frac{1}{m_2(L_t)^{1/2}} \left\{ \frac{\sqrt{8}}{\varepsilon(m_0(L_t) - \eta)} + e^{-t(\frac{8}{3\pi} - \frac{3}{4})} \right\}. \end{aligned}$$

The last upper bound, in turn, coincides with ε if

$$\varepsilon = \varepsilon_t := \frac{g_2(t) + \sqrt{g_2(t)^2 + g_1(t)}}{m_2(L_t)^{1/4}}.$$

Thus, the Ky Fan distance between X and O does not exceed ε_t and the thesis of Theorem 2 follows from the well-known fact that this distance dominates the Prokhorov one. See, e.g., Theorem 11.3.5 in Dudley (2002).

To prove Corollary 3, it is enough to replace the random variable $\tilde{\nu}_t$ with numbers ν_n and the random variables π_l with numbers $\lambda_{j,n}$ and, then, to proceed as in the previous proofs of Theorems 1 and 2.

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