Algebraic rate of decay for the excess free energy and stability of fronts for a non–local phase kinetics equation with a conservation law, II

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Abstract We continue our study of a non–local evolution equation that describes the evolution of the local magnetization in a continuum limit of an Ising spin system with Kawasaki dynamics and Kac potentials. We consider sub–critical temperatures, for which there are two local equilibria, and complete the proof of a local nonlinear stability result for the minimum free energy profile for the magnetization at the interface between regions of these two different local equilibrium; i.e., the fronts. We show that an initial perturbation v_0 of a front that is sufficiently small in L^2 norm, and sufficiently localized that $\int x^2 v_0(x)^2 \mathrm{d}x < \infty$, yields a solution that relaxes to another front, selected by a conservation law, in the L^1 norm at an algebraic rate that we explicitly estimate. We also obtain rates for the relaxation in the L^2 norm, and the rate of decrease of the excess free energy.

Introduction

We continue our study of the nonlocal and nonlinear evolution equation

$$\frac{\partial}{\partial t} m(x,t) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} m(x,t) - \beta(1-m(x,t)^2)\left(J \star \frac{\partial}{\partial x} m\right)(x,t)\right) \tag{1.1}$$

begun in [3]. Further background on this equation, introduced in [14], is contained in [3], and we shall be very brief on such matters here. Instead, we shall focus on the points most relevant to the analysis to be done here, and rely on familiarity with [3] for the rest.

First, the free energy functional $\mathcal{F}(m)$

$$\mathcal{F}(m) = \int_{I\!R^n} [f(m(x)) - f(m_\beta)]\mathrm{d}x + \frac{1}{4}\int_{I\!R^n}\int_{I\!R^n} J(x-y)[m(x)-m(y)]^2 \mathrm{d}x\mathrm{d}y \tag{1.2}$$

with

$$f(m) = -\frac{1}{2}m^2 + \frac{1}{\beta}\left[\left(\frac{1+m}{2}\right)\ln\left(\frac{1+m}{2}\right) + \left(\frac{1-m}{2}\right)\ln\left(\frac{1-m}{2}\right)\right] \tag{1.3}$$

is a Lyapunov functional for this equation. For $\beta > 1$, the potential function f is a symmetric double well potential on $[-1,1]$. We denote the positive minimizer of f on $[-1,1]$ by m_β.

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Then the equation (1.1) can be written as

$$\frac{\partial}{\partial t}m = \frac{\partial}{\partial x}\left(\sigma(m)\frac{\partial}{\partial x}\left(\frac{\delta \mathcal{F}}{\delta m}\right)\right) \tag{1.4}$$

where the *mobility* $\sigma(m)$ is given by

$$\sigma(m) = \beta(1-m^2) \tag{1.5}$$

from which it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(m(t)) = -\int\left|\frac{\partial}{\partial x}\left(\frac{\delta \mathcal{F}}{\delta m}\right)\right|^2\sigma(m)\mathrm{d}x = -\mathcal{I}((m(t)) \tag{1.6}$$

where the equality on the right defines the functional $\mathcal{I}(m)$.

The fronts studied here are the profiles that minimize the free energy while intepolating between the two equilibrium values $\pm m_\beta$. It has been shown in [11] that there is a unique function $\bar{m}_0(x)$ such that

$$\mathcal{F}(\bar{m}_0) = \inf\left\{\ \mathcal{F}(m)\ \middle|\ \mathrm{sgn}(x)m(x) \geq 0,\ \lim_{x\to\pm\infty}\mathrm{sgn(x)m(x)} > 0\right\} \tag{1.7}$$

Furthermore it is shown that $\bar{m}_0$ is an odd increasing function, and that

$$\begin{array}{lll} 0 < m_\beta^2 - \bar{m}_0^2(x) & \leq Ce^{-\gamma|x|} \\ 0 < \quad \bar{m}_0'(x) & \leq Ce^{-\gamma|x|} \\ 0 < \quad |\bar{m}_0''(x)| & \leq Ce^{-\gamma|x|} \end{array} \tag{1.8}$$

for positive constants C and γ depending on J and β. The first two of these estimates are proved in [11] and the third one is proved in [7].

The subscript 0 on the minimizer refers to the fact that the constraint imposed in (1.7) breaks the translational invariance of the free energy. For any a in $I\!R$, define

$$\bar{m}_a(x) = \bar{m}_0(x-a)\ . \tag{1.9}$$

These functions $\bar{m}_a$ are the fronts whose stability is to be investigated here. Clearly $\mathcal{F}(\bar{m}_a) = \mathcal{F}(\bar{m}_0)$, so that $\bar{m}_0$ belongs to a one parameter family of minimizers of the free energy. There is another family, obtained by reflecting the previous one, because the free energy is also reflection invariant. However, it is the fact that there is a continuous family of minimizers – due to the translation invariaince of $\mathcal{F}$ – that weakens the local dissipativity of our problem in a crucial way.

Now, besides having a Lyapunov function, the equation (1.1) has a conservation law. Indeed, for any b,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\big(m(x,t) - \bar{m}_b(x)\big)\mathrm{d}x = 0\ . \tag{1.10}$$

Therefore, if one defines a in terms of initial data m_0 for (1.1) by

$$\int\big(m(x,0) - \bar{m}_a(x)\big)\mathrm{d}x = 0\ , \tag{1.11}$$

one has for the solution

$$\int\big(m(x,t) - \bar{m}_a(x)\big)\mathrm{d}x = 0 \tag{1.12}$$

for all t. (Note that clearly there is just one value of a for which (1.11) holds.)

It is now clear what should happen if we solve (1.1) for initial data $m_0(x)$ that is a small perturbation of the centered front $\bar{m}_0(x)$: The decrease of the excess free energy should force the evolving profile to tend to the family of fronts, and then the conservation law should determine the particular front it tends to, and one expects

$$\lim_{t\to\infty} m(x,t) = \bar{m}_a(x) \tag{1.13}$$

with a given in terms of m_0 by (1.11).

We do in fact prove this and more here. The main result of this paper is the following, which has been announced in [3]:

Theorem 1.1 *Consider initial data $m_0(x)$ for (1.1) such that*

$$\int x^2 (m_0(x) - \bar{m}_0(x))^2 \mathrm{d}x \le c_0 \ , \tag{1.14}$$

where c_0 is any positive constant. Then for any $\delta > 0$ there is a strictly positive constant $\epsilon = \epsilon(\delta, c_0, \beta, J)$ depending only on δ, c_0, β and J such that for all inital data m_0 with $-1 \le m_0 \le 1$, and with

$$\int (m_0(x) - \bar{m}_0(x))^2 \mathrm{d}x \le \epsilon \ , \tag{1.15}$$

the excess free energy $\mathcal{F}(m(t)) - \mathcal{F}(m_0)$ of the corresponding solution $m(t)$ of (1.1) satisfies

$$\mathcal{F}(m(t)) - \mathcal{F}(\bar{m}) \le c_2 (1 + c_1 t)^{-(9/13 - \delta)} \tag{1.16}$$

and

$$\|m(t) - \bar{m}_a\|_1 \le c_2 (1 + c_1 t)^{-(5/52 - \delta)} \tag{1.17}$$

where c_1 and c_2 are finite constants depending only on δ, c_0, J and β and a is given by (1.11).

However, the proof is not so simple as one might hope, based on the heuristics discussed before the theorem. There are several reasons for this. The first has to do with the relevant norms.

To explain the physical relevance of the L^2 norm in (1.15), first recall that it has been shown in [3] that there is a value of b so that

$$\|m - \bar{m}_b\|_2 = \inf_{c\in \mathbb{R}} \{\|m - \bar{m}_c\|_2\} \ ,$$

and moreover, when $\|\bar{m}_b - m\|_2$ is sufficiently small, this minimum is attained *uniquely* at b.

Lemma 1.2 of [3] then says that the excess free energy measures the distance to this closest front in the L^2 metric in the sense that for any $\kappa > 0$, there are constants $\delta = \delta(\kappa) > 0$ and $C = C(\kappa) < \infty$ such that

$$\frac{1}{C} \|m - \bar{m}_b\|_2^2 \le \mathcal{F}(m) - \mathcal{F}(\bar{m}_b) \le C \|m - \bar{m}_b\|_2^2 \tag{1.18}$$

whenever $\|(m - \bar{m}_b)'\|_2 \le \kappa$, $\|m - \bar{m}_b\|_2 \le \delta$, and $\bar{m}_b$ is any front that minimizes $\|m - \bar{m}_b\|_2$. As in [3], we use the smoothing properties of (1.1) to obtain the condition on $\bar{m}'(t)$ for all $t \ge t_0$, some finite t_0 given by Lemma 2.2 of [3].

On account of (1.18), for any solution $m(t)$ of (1.1), define $a(t)$ to be that value of b such that

$$\|m(t) - \bar{m}_{a(t)}\|_2 = \inf_{b\in \mathbb{R}} \{\|m(t) - \bar{m}_b\|_2\} \tag{1.19}$$

and note that $a(t)$ is a well–defined function as long as $\|m(t) - \bar{m}_{a(t)}\|_2$ stays sufficiently small since then the minimum is uniquely attained. (We shall do all of our analysis in this paper for times t in an interval (t_0, T_0) on which $\|m(t) - \bar{m}_{a(t)}\|_2$ does stay small, and then at the end we shall show that $T_0 = \infty$.)

Hence, if one proves that the excess free energy decreases to zero, the best one can obtain from this is that

$$\lim_{t\to\infty} \|m(t) - \bar{m}_{a(t)}\|_2 = 0$$

However, this doesn't yield any information on $a(t)$ – and it cannot by the translation invariance of the free energy.

The conservation law would give us information on $a(t)$, but to use it we require L^1 control on $m(\cdot,t) - \bar{m}_{a(t)}(\cdot)$. Since

$$\|m(\cdot,t) - \bar{m}_{a(t)}(\cdot)\|_\infty \le 2$$

a–priori, L^1 control would give us L^2 control through

$$\|m(\cdot,t) - \bar{m}_{a(t)}(\cdot)\|_2^2 \le 2\|m(\cdot,t) - \bar{m}_{a(t)}(\cdot)\|_1$$

but not *vice–versa.* In order to use the conservation law to show that

$$\lim_{t\to\infty} a(t) = a \tag{1.20}$$

where a is given by (1.11), we must, and shall, show that

$$\lim_{t\to\infty} \|m(\cdot,t) - \bar{m}_{a(t)}\|_1 = 0\ .$$

Moreover, as explained in [3], one needs something comparable to L^1 control even to show that the excess free energy does decrease all the way to zero. The mere fact that the derivative is negative – all that (1.6) says for free – does not imply even this.

Before discussing the L^1 behavior of perturbations of fronts, we make the following convention, to be used throughout the paper, whenever some solution $m(x,t)$ is under discussion:

$$v(x,t) = m(x,t) - \bar{m}_{a(t)}(x) \tag{1.21}$$

where $a(t)$ is given in (1.19), and moreover

$$\bar{m}(x) \qquad \text{denotes} \qquad \bar{m}_{a(t)}(x). \tag{1.22}$$

This same convention was imposed in [3].

One of the main results of that paper, Theorem 3.2, was a lower bound on the rate of dissipation of the excess free energy: For any $\epsilon > 0$,

$$\frac{d}{dt}\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar{m}\big)\big] = -\mathcal{I}(m(t)) \le -(1-\epsilon)\int \sigma(\bar{m}(x))\,[(\mathcal{A}v)'(x)]^2\,\mathrm{d}x \tag{1.23}$$

whenever $||v'|| \le \kappa_1(\beta,J,\epsilon)$ and $||v|| \le \delta_1(\beta,J,\epsilon)$ for some strictly positive constants $\kappa_1(\beta,J,\epsilon)$ and $\delta_1(\beta,J,\epsilon)$. Here $\mathcal{A}$ denotes the second variation of the free energy $\mathcal{F}$ at $\bar{m}$. By our convention, $\bar{m}$ denotes $\bar{m}_{a(t)}$, and while it is occasionally preferable to write $\mathcal{A}_{a(t)}$ to make this explicit, we shall generally simply write $\mathcal{A}$, and leave the dependence on $a(t)$ implicit. However, in recalling the definition, we shall be explicit:

$$\langle u, \mathcal{A}_a u\rangle_{L^2} = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{F}(\bar{m}_a + su)\bigg|_{s=0} \tag{1.24}$$

A number of properties of $\mathcal{A}$ that we shall freely use in our analysis here are discussed in [3], which we assume to be familiar.

Because of the derivatives, the quadratic form on the right in (1.23) has no spectral gap. If it did, this together with (1.18) would provide an exponential rate of decrease of the excess free energy, and hence of $\|v(t)\|_2$. This appraoch was used used in [5] for the corresponding problem with a non conservative dynamics, for which there is a spectral gap. However, since there is no spectral gap here, one needs additional monotonicity, or at least *a–priori* boundedness properties to exploit (1.23), as explained in [3]. In the study of parabolic equations

$$\frac{\partial u}{\partial t} = \nabla \cdot \big(D(u)\nabla u\big) \ , \tag{1.25}$$

for which there is also no spectral gap,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |u(x,t)|\mathrm{d}x \le 0 \tag{1.26}$$

which trivially provides the additional monotonicity required to show that

$$\sup_{t\ge 0} \|u(t)\|_1 \le \|u(0)\|_1 \ .$$

Then, as explained in [3], a standard argument with the Nash inequality allows one to conclude that decreases to zero at an algebraic rate, at least when the diffusivity $D(\cdot)$ in (1.25) is bounded from below.

This route is closed to us since the analog of (1.26) does not hold [3] for $v(t)$ when $m(t)$ is a solution to (1.1). Moreover, there are other problematic non-dissipative features, namely, the maximum principle fails to hold for (1.1). Since the mobility (1.5) vanishes where $m = \pm 1$, and with it some part of the dissipation in (1.6), one needs something else to keep the solutions away from ± 1.

In this paper, we refine the analysis made in [3] using the "uncertainty principle"; i.e.,

$$\left(\int x^2|\psi(x)|^2\mathrm{d}x\right)\left(\int |\psi'(x)|^2\mathrm{d}x\right) \ge \frac{1}{4}\left(\int |\psi(x)|^2\mathrm{d}x\right)^2 \tag{1.27}$$

to obtain bounds on the decay in the L^1 norm, instead of simply the L^2 norm as in [3].

To illustrate this, we again turn to the heat equation. Consider a solution $u(x,t)$ of the heat equation

$$\frac{\partial}{\partial t}u(x,t) = u''(x,t)$$

with integrable initial data u_0, and suppose that

$$\int u_0(x)\mathrm{d}x = 0 \ . \tag{1.28}$$

Then

$$\int u(x,t)\mathrm{d}x = 0 \tag{1.29}$$

for all t. Define

$$\int |u(x,t)|^2\mathrm{d}x \qquad \text{and} \qquad \phi(t) = \int x^2|u(x,t)|^2\mathrm{d}x$$

and, as in [3], one finds

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \le 2f(t) \ . \tag{1.30}$$

However, the condition (1.28) allows us to make use of stronger uncertainty principle estimate. It is shown in Theorem 2.1 of this paper that under the constraint

$$\int \psi(x)\mathrm{d}x = 0$$

one has

$$\left(\int x^2|\psi(x)|^2\mathrm{d}x\right)\left(\int |\psi'(x)|^2\mathrm{d}x\right) \geq \frac{9}{4}\left(\int |\psi(x)|^2\mathrm{d}x\right)^2 \tag{1.31}$$

Hence, applying (1.31),

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) = -2\int |u'(x,t)|^2\mathrm{d}x \leq$$
$$-\frac{9}{2}\left(\int |u'(x,t)|^2\mathrm{d}x\right)^2\left(\int |xu(x,t)|^2\mathrm{d}x\right)^{-1} = -\frac{9}{2}\frac{f^2(t)}{\phi(t)}$$

Therefore we have the system of differential inequalities

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}f(t) &\leq -A\frac{f(t)^2}{\phi(t)}\\ \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) &\leq Bf(t)\end{aligned} \tag{1.32}$$

with $A = 9/2$ and $B = 2$. Theorem 5.1 of [3] says that for any solution of (1.32),

$$f(t) \leq f(0)^{1-q}\phi(0)^q\left(\frac{\phi(0)}{f(0)} + (A+B)t\right)^{-q}$$
$$\phi(t) \leq f(0)^{1-q}\phi(0)^q\left(\frac{\phi(0)}{f(0)} + (A+B)t\right)^{1-q}$$

where

$$q = \frac{A}{A+B}\ .$$

In the case at hand, this is

$$q = \frac{9}{13}\ .$$

Since this value exceeds 1/2, we get L^1 decay in the following way: We prove in Lemma 5.2 of the present paper that for any function w and any $0 < \delta < 1$

$$\|w\|_1 \leq C(\delta)\|(1+x^2)^{1/2}w\|_2^{(1+\delta)/2}\|w\|_2^{(1-\delta)/2}$$

where $C(\delta)$ is a finite constant given explicitly in the lemma. Since $9/13 > 1/2$ for δ sufficiently small, we have that $\|(1+x^2)^{1/2}u(t)\|_2^{(1+\delta)/2}$ increases more slowly than $\|w\|_2^{(1-\delta)/2}$ increases, and so $\|u(t)\|_1$ decreases to zero. In fact, the rate one gets is arbitrarily close to $t^{-5/26}$, for δ sufficiently small, as in Theorem 1.1.

Here, to prove stability for (1.1), we define

$$f(t) = \mathcal{F}(\bar{m}+v(t)) - \mathcal{F}(\bar{m}) \qquad \text{and} \qquad \phi(t) = 1 + \int \sigma(\bar{m})x^2|\mathcal{A}v(x,t)|^2\mathrm{d}x\ . \tag{1.33}$$

The definition of ϕ in (1.33) differs from the definitions made in [3] only in the addition of 1, which ensures that $\phi(t) \geq 1$.

We have estimated the time derivatives of these quantities in [3], and obtained bounds of the form given in (1.32), but with inexplicit constants A and B.

Now, the rate of decay that one gets by this method depends very much on the ratio of the constants A and B in (1.32). To get L^1 decay, we need this ratio to be fairly close to the ratio 9/13 obtained for the heat equation. (Actually, one can do better for the heat equation; see [3].)

We do this by exploiting the following alternatives: for any $\epsilon_1 > 0$, at any time t, one has either

$$\mathcal{I}(m(t)) \leq \epsilon_1 \bigl[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})\bigr] \ . \tag{1.34}$$

or

$$\mathcal{I}(m(t)) \geq \epsilon_1 \bigl[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})\bigr] \ . \tag{1.35}$$

where $\mathcal{I}$ is the dissipation functional (1.6).

We prove in section 3 that for any $\epsilon > 0$, there are strictly positive constants $\delta_0(\beta, J, \epsilon)$, $\kappa_0(\beta, J, \epsilon)$ and $\epsilon_1(\beta, J, \epsilon)$ depending only on β, J and ϵ, such that for all t for which (1.34) is satisfied together with $||v'(t)||_2 < \kappa_0$, $||v(t)||_2 < \delta_0$ and $|a(t)| \leq 1$, it is the case that

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigl[\mathcal{F}\bigl(m(t)\bigr) - \mathcal{F}\bigl(\bar{m}\bigr)\bigr] \leq -9(1-\epsilon)(1-\sigma(m_\beta))^2 \frac{\bigl[\mathcal{F}\bigl(m(t)\bigr) - \mathcal{F}(\bar{m})\bigr]^2}{\phi(t)} \ .$$

We then show in section 4 that under the same assumptions of section 3, it is the case that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \leq (1+\epsilon)4(1-\sigma(m_\beta))^2 \bigl[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})\bigr] \ .$$

Notice the condition that $|a(t)| \leq 1$, to which we shall return. Thus, when (1.34) holds, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f(t) &\leq -\tilde{A}\frac{f(t)^2}{\phi(t)} \\ \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) &\leq \tilde{B} f(t) \end{aligned} \tag{1.36}$$

with the difference between $\tilde{A}/\tilde{B}$ and 9/13 arbitrarily small for ϵ small enough for all times t such that $\|v(t)\|_2$, $\|v'(t)\|_2$ are sufficiently small and $|a(t)| \leq 1$.

On the other hand, when (1.34) is violated and (1.35) holds, the dissipation is large, and this works in our favor. In section 5 of the paper, we exploit this alternative to prove Theorem 1.1. The proof is still somewhat intricate, and it would have been simplified had we been able to show the existence of a time $t_\star$ such that (1.34) holds for all $t \geq t_\star$. If this were the case, the constants $\tilde{A}$ and $\tilde{B}$ above would govern the decay, and we would obtain a bound on the excess fee energy of the form

$$\bigl[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})\bigr] \leq C(1 + D(1-\sigma(m_\beta))^2 t)^{-q}$$

where D does not depend on β. Since $(1-\sigma(m_\beta))^2$ vanishes as the critical temperature is approached, and $\tilde{\alpha}$ vanishes as β approaches 1, this would indicate how the rate of relaxation slows in this limit. In any case, our results do show that it is posible to estimate the *exponent* in the rate of relaxation independently of β.

To explain why (1.34) enables us to obtain what are essentially heat equation constant in (1.36), one has to view it as a smoothness condition. Indeed, as we explain is section 3 here, it follows from Theorem 3.2 of [3] that

$$\sigma(m_b)\|\big(\mathcal{A}v\big)'\|_2^2 \le (1-\epsilon)\mathcal{I}(\bar{m}+v)$$

for any ϵ, under appropriate conditions on v, and hence, by Lemma A.2 of [3], which compare $\|\mathcal{A}v\|_2^2$ and the excess free energy of $\bar{m}+v$, when (1.34) holds,

$$\|\big(\mathcal{A}v\big)'\|_2^2 << \|\mathcal{A}v\|_2^2 \ . \tag{1.37}$$

Next, the action of $\mathcal{A}$ on functions w that satisfy

$$\|w'\|_2 << \|w\|_2$$

is particularly simple: As shown in section 2,

$$\mathcal{A}w \approx \tilde{\alpha}w$$

where $\tilde{\alpha} = 1/\sigma(m_\beta) - 1$ and the error is small percentagewise in the L^2 norm. Once one may replace $\mathcal{A}$ with multiplication by α, the linearized version of (1.1) does become essentially the heat equation. This discussion is heuristic, but in no way missleading, and hopefully motivates the technical preliminaries in section 2.

We have by now described the structure of the paper, and turn to section 2. This contains a number of preliminary estimates, including a proof of the constrained uncertainty principle (1.31).

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2 Preliminary Estimates

The first result presented in this section is a constrained version of Weyl's uncertainty principle inequality:

Theorem 2.1 *Let $\psi(x)$ be a function on the real line such that*

$$\int |\psi'(x)|^2 \mathrm{d}x < \infty \qquad \text{and} \qquad \int |x\psi(x)|^2 \mathrm{d}x < \infty \tag{2.1}$$

and such that either

$$\psi(0) = 0 \tag{2.2}$$

or

$$\int \psi(x) \mathrm{d}x = 0\ . \tag{2.3}$$

Then

$$\Big(\int |\psi'(x)|^2 \mathrm{d}x\Big)\Big(\int |x\psi(x)|^2 \mathrm{d}x\Big) \geq \frac{9}{4}\Big(\int |\psi(x)|^2 \mathrm{d}x\Big)^2 \tag{2.4}$$

First notice that under (2.1), ψ is integrable and well–defined at 0, so (2.2) and (2.3) make sense.

Proof: It suffices to prove that

$$\mathcal{Q}(\phi) \geq (3/2)\|\phi\|_2^2 \tag{2.5}$$

for all ϕ satisfying (2.2), where $\mathcal{Q}$ is the quadratic form given by

$$\mathcal{Q}(\phi) = \int |\phi'(x)|^2 \mathrm{d}x + \frac{1}{4}\int x^2|\phi(x)|^2 \mathrm{d}x \tag{2.6}$$

This is because if we rescale such a function $\phi(x)$ by defining $\phi_{(\kappa)}(x) = (\kappa)^{1/2}\phi(\kappa x)$, so that $\|\phi_{(\kappa)}\|_2 = \|\phi\|_2$, and insert this in (2.6), we would then get

$$\mathcal{Q}(\phi_{(\kappa)}) = \kappa \int |\phi'(x)|^2 \mathrm{d}x + \kappa^{-1}\frac{1}{4}\int x^2|\phi(x)|^2 \mathrm{d}x \tag{2.7}$$

after a change of variables. Now minimizing over κ in (2.7), one obtains

$$\inf_{\kappa}\{\ \mathcal{Q}(\phi_{(\kappa)})\} = \Big(\int |\phi'(x)|^2 \mathrm{d}x\Big)^{1/2}\Big(\int |x\phi(x)|^2 \mathrm{d}x\Big)^{1/2}\ . \tag{2.8}$$

However, (2.2) is invariant under scaling, and hence $\phi_{(\kappa)}(0) = 0$ for all κ. Thus, one obtains (2.4) from (2.8) and (2.5).

To prove (2.5) for functions ϕ satisfying (2.2), define

$$\phi_{\pm} = \mathrm{sgn(x)}\phi(\pm|x|)$$

so that both ϕ_- and ϕ_+ are antisymmetric functions satisfying the conditions (2.1) because of (2.2). Next, note that

$$\mathcal{Q}(\phi) = \frac{1}{2}\big(\mathcal{Q}(\phi_-) + \mathcal{Q}(\phi_+)\big)\ . \tag{2.9}$$

Now, for any function ϕ,

$$\mathcal{Q}(\phi) = \langle \phi, H\phi \rangle_{L^2}$$

where H is the harmonic oscillator Hamiltonian

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{4}x^2 \ .$$

This has simple eigenvalues $E_k = k + 1/2$ for all non-negative integers k. The corresponding eigenfunctions are the Hermite functions $h_k(x)$ which are even or odd according to the value of k. Hence the smallest eigenvalue of H in the odd subspace is $3/2$, so that

$$\mathcal{Q}(\phi_\pm) \geq \frac{3}{2}\|\phi_\pm\|_2^2 \ .$$

clearly holds. From this and (2.9), (2.5) clearly follows.

Though we do not use (2.3) here, it is worth noting in passing that if ψ satisfies (2.3), then its Fourier transform

$$\hat{\psi}(k) = \int e^{2\pi i x k}\psi(x)\mathrm{d}x$$

satisfies (2.2). The result then follows from the invariance of the family of quadratic forms in (2.7) under the Fourier transform. ■

The approach in this paper makes essential use of the smoothness that develops over time in solutions of diffusive equations. The following simple lemma will be used here in a number of ways to exploit this smoothness.

Lemma 2.2 *Let $\rho(x)$ be a probability density with*

$$\int |x|\rho(x)\mathrm{d}x < \infty \ .$$

Then for any square integrable function w with square integrable first derivative w',

$$\|w - \rho\star w\|_2 \leq \left(\frac{1}{2\pi^2}\int |x|\rho(x)\mathrm{d}x\right)\|w'\|_2 \ . \tag{2.10}$$

Proof: Fourier transforming,

$$\begin{aligned}&\|w - \rho\star w\|_2^2 = \\ &\int |\hat{w}(k)|^2|1-\hat{\rho}(k)|^2\mathrm{d}k \leq \\ &\left\|\frac{|1-\hat{\rho}(k)|^2}{|2\pi k|^2}\right\|_\infty \int |\hat{w}(k)|^2|2\pi k|^2\mathrm{d}k = \\ &\left\|\frac{|1-\hat{\rho}(k)|^2}{|2\pi k|^2}\right\|_\infty \|w'\|_2^2\end{aligned}$$

To conclude the proof, note that

$$|\hat{\rho}(k) - 1| \leq 2|k| \int |x|\rho(x)\mathrm{d}x \ .$$

■

The next lemma shows that for any function w that is orthogonal to $\bar{m}'$, whenever $\|w'\|_2$ is small compared to $\|w\|$, then $\mathcal{A}w$ is very close to being a constant multiple of w, $\tilde{\alpha}w$ where $\tilde{\alpha}$ is defined by

$$\tilde{\alpha} = \frac{1}{\beta(1 - m_\beta^2)} - 1 \tag{2.11}$$

and it is strictly positive for $\beta > 1$. The lemma also shows that under the same condition, $\sigma(\bar{m})w$ is very close to $\sigma(m_\beta)w$.

Lemma 2.3 *There is a finite constant $K(\beta, J)$ depending only on β and J so that*

$$\|Aw - \tilde{\alpha}w\|_2 \leq K(\beta, J)\|w'\|_2 \ . \tag{2.12}$$

and

$$\|\sigma(\bar{m})w - \sigma(m_\beta)w\|_2 \leq K(\beta, J)\|w'\|_2 \ . \tag{2.13}$$

for all functions w with $\langle w, \bar{m}'\rangle_{L^2} = 0$.

Proof: First, for any y and x we have

$$w(x) = w(y) + \int_y^x w'(z)\mathrm{d}z \ .$$

Now multiply both sides by $\bar{m}'(y)$ and integrate in y. By the orthogonality of $\bar{m}'$ and w, we have

$$2m_\beta w(x) = \int_{-\infty}^{\infty} \bar{m}'(y)\left(\int_y^x w'(z)\mathrm{d}z\right)\mathrm{d}y \ .$$

But $|\int_y^x w'(z)\mathrm{d}z| \leq |x-y|^{1/2}\|w'\|_2$ so that

$$|w(x)| \leq \frac{1}{2m_\beta}\left(\int \bar{m}'(y)|x-y|^{1/2}\mathrm{d}y\right)\|w'\|_2 \ ,$$

and clearly there is a finite constant $K(\beta, J)$ depending only on β and J so that

$$\frac{1}{2m_\beta}\int \bar{m}'(y)|x-y|^{1/2}\mathrm{d}y \leq K(\beta, J)(1 + |x|),$$

and hence

$$|w(x)| \leq K(\beta, J)(1 + |x|)\|w'\|_2 \ . \tag{2.14}$$

Next, consider (2.12). Clearly,

$$Aw - \tilde{\alpha}w = \frac{1}{\beta}\left(\frac{\bar{m}^2 - m_\beta^2}{(1 - \bar{m}^2)(1 - m_\beta^2)}\right)w + (w - J \star w)$$

We will estimate these two terms separately. First note that Lemma 2.2 takes care of $(w - J \star w)$. Next, using the pointwise bounds (2.14) established above,

$$
\begin{aligned}
&\|(\bar{m}^2 - m_\beta^2)\big((1-\bar{m}^2)(1-m_\beta^2)\big)^{-1} w\|_2^2 \le \\
&\|w'\|_2^2 K(\beta, J) \int (1+|x|)^2 (\bar{m}^2 - m_\beta^2)^2 \big((1-\bar{m}^2)(1-m_\beta^2)\big)^{-2} \mathrm{d}x \le \\
&\tilde{K}(\beta, J)\|w'\|_2^2
\end{aligned}
$$

where $\tilde{K}(\beta, J)$ is finite by the rapid decay of $(\bar{m}^2 - m_\beta^2)^2$. Drop the tilde from $\tilde{K}(\beta, J)$, and this establishes (2.12). The proof of (2.13) is very similar to the proof of (2.12). ∎

3 Dissipation rate of the Free Energy

In this section we establish a bound on the rate $\mathcal{I}(m(t))$ at which the excess free energy $\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})$ is dissipated in term of $\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})$ itself, working under the hypothesis that

$$\mathcal{I}(m(t)) << \left[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})\right] . \tag{3.1}$$

This additional information permits much tighter estimates than were obtained in [3], and on the other hand, when (3.1) is not satisfied, there is ample dissipation, as explained in the introduction. The main result of this section is the following.

Theorem 3.1 *Let $m(\cdot, t)$ be a solution of (1.1). Then for any $\epsilon > 0$, there are strictly positive constants $\delta_0(\beta, J, \epsilon)$, $\kappa_0(\beta, J, \epsilon)$ and $\epsilon_1(\beta, J, \epsilon)$ depending only on β, J and ϵ, such that for all t with*

$$\mathcal{I}(m(t)) \leq \epsilon_1 \left[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})\right] \tag{3.2}$$

$||v'(t)||_2 < \kappa_0$, $||v(t)||_2 < \delta_0$ and $|a(t)| \leq 1$, it is the case that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar{m}\big)\right] \leq -9(1-\epsilon)(1-\sigma(m_\beta))^2 \frac{\left[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar{m}\big)\right]^2}{\phi(t)} . \tag{3.3}$$

Proof: We know from Theorem 3.2 of [3] that

$$\frac{d}{dt}\left[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar{m}\big)\right] = -\mathcal{I}(m(t)) \leq -(1-\epsilon)\int \sigma(\bar{m}(x))\left[(\mathcal{A}v)'(x)\right]^2 \mathrm{d}x \tag{3.4}$$

whenever $||v'|| \leq \kappa_1(\beta, J, \epsilon)$ and $||v|| \leq \delta_1(\beta, J, \epsilon)$ for some strictly positive constants $\kappa_1(\beta, J, \epsilon)$ and $\delta_1(\beta, J, \epsilon)$.

Next, since $|\bar{m}| \leq m_\beta$ everywhere,

$$\sigma(\bar{m}) \geq \beta(1 - m_\beta^2) = \sigma(m_\beta)$$

and hence (3.4) implies

$$\mathcal{I}(m(t)) \geq (1-\epsilon)\sigma(m_\beta)\int \left[(\mathcal{A}v)'(x)\right]^2 \mathrm{d}x . \tag{3.5}$$

The argument that follows is slightly intricate. To keep the thread from getting lost, we first suppose that the initial data is antisymmetric, and hence that $\mathcal{A}v(0) = 0$. This permits application of the conditional uncertainty principle, and one obtains

$$\mathcal{I}(m(t)) \geq (1-\epsilon)\sigma(m_\beta)\frac{9}{4}\frac{\|\mathcal{A}v\|_2^4}{\|x\mathcal{A}v\|_2^2} \tag{3.6}$$

We shall show below that we can obtain essentially the same result using the orthogonality condition

$$\langle \bar{m}', \mathcal{A}v\rangle_{L^2} = 0 .$$

This is because if we define $\rho(x) = (2m_\beta)^{-1}\bar{m}_0'(x)$, then

$$\rho \star \big(\mathcal{A}v\big)(a(t)) = \langle \bar{m}', \mathcal{A}v\rangle_{L^2} = 0$$

on the one hand, and on the other hand

$$\rho \star \big(\mathcal{A}v\big) \approx \mathcal{A}v$$

by Lemma 2.2.

We shall return to this after first proceeding to analyse (3.6). First, since $\mathcal{A}$ is self adjoint,

$$\begin{aligned}\|\mathcal{A}v\|_2^2 =&\langle v, \mathcal{A}^2 v\rangle_{L^2} = \\ &\tilde{\alpha}\langle v, \mathcal{A}v\rangle_{L^2} + \langle v, (\mathcal{A}-\tilde{\alpha})\mathcal{A}v\rangle_{L^2}\end{aligned} \tag{3.7}$$

and hence, by Lemma A.1 of [3],

$$\|\mathcal{A}v\|_2^2 \geq 2(1-\epsilon)\tilde{\alpha}\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar{m}\big)\big] - |\langle v, (\mathcal{A}-\tilde{\alpha})\mathcal{A}v\rangle_{L^2}| \ . \tag{3.8}$$

We now need to show that

$$|\langle v, (\mathcal{A}-\tilde{\alpha})\mathcal{A}v\rangle_{L^2}| \leq \epsilon\tilde{\alpha}2\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar{m}\big)\big] \ . \tag{3.9}$$

At this point, (3.2) enters the argument.

Observe that by (3.5) and (3.2),

$$\|(\mathcal{A}v)'\|_2^2 \leq \frac{\epsilon_1}{(1-\epsilon)\sigma(m_\beta)}\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar{m}\big)\big] \ . \tag{3.10}$$

But by Lemma 2.3 and then by the Schwarz inequality,

$$\begin{aligned}|\langle v, (\mathcal{A}-\tilde{\alpha})\mathcal{A}v\rangle_{L^2}| \leq&\|v\|_2\|(\mathcal{A}-\tilde{\alpha})\mathcal{A}v\|_2 \leq \\ &K(\beta,J)\|v\|_2\|(\mathcal{A}v)'\|_2 \leq \\ &K(\beta,J)\|v\|_2\bigg(\frac{\epsilon_1}{(1-\epsilon)\sigma(m_\beta)}\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar{m}\big)\big]\bigg)^{1/2}\end{aligned}$$

where we have used (3.10) in the last step. Finally, by the spectral gap inequality for $\mathcal{A}$ and Lemma A.1 of [3] once again, one has

$$\|v\|_2^2 \leq \frac{1}{\alpha}\langle v, \mathcal{A}v\rangle_{L^2} \leq 2(1+\epsilon)\frac{1}{\alpha}\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar{m}\big)\big]$$

and hence

$$|\langle v, (\mathcal{A}-\tilde{\alpha})\mathcal{A}v\rangle_{L^2}| \leq C(\beta,J)\bigg(\frac{\epsilon_1}{\alpha(1-\epsilon)\sigma(m_\beta)}\bigg)^{1/2}\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar{m}\big)\big]$$

for some constant $C(\beta,J)$ depending only on β and J. Now choosing ϵ_1 so that

$$\frac{C(\beta,J)}{\tilde{\alpha}}\bigg(\frac{\epsilon_1}{\alpha(1-\epsilon)\sigma(m_\beta)}\bigg)^{1/2} \leq \epsilon$$

one has from (3.8)

$$\|\mathcal{A}v\|_2^2 \geq 2(1-2\epsilon)\tilde{\alpha}\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar{m}\big)\big] \ . \tag{3.11}$$

We record for future use that essentially the same argument under exactly the same hypotheses yields the bound

$$\|\mathcal{A}v\|_2^2 \leq 2(1+2\epsilon)\tilde{\alpha}\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar{m}\big)\big] \ . \tag{3.12}$$

Combining (3.11) and (3.6) yields

$$\mathcal{I}(m(t)) \geq 9(1-2\epsilon)^2\sigma(m_\beta)\tilde{\alpha}^2\frac{\big[\mathcal{F}\big(m(t)\big)-\mathcal{F}\big(\bar{m}\big)\big]^2}{\|x\mathcal{A}v\|_2^2}\ .$$

The last step, under our current assumptions, is to estimate $\|x\mathcal{A}v\|_2^2$ in terms of $\phi(t)$. This is easily done since

$$\begin{aligned}&\sigma(m_\beta)\|x\mathcal{A}v\|_2^2 = \\ &\int\sigma(\bar{m})x^2\big(\mathcal{A}v\big)^2\mathrm{d}x + \int\big(\sigma(m_\beta)-\sigma(\bar{m})\big)x^2\big(\mathcal{A}v\big)^2\mathrm{d}x \leq \\ &\int\sigma(\bar{m})x^2\big(\mathcal{A}v\big)^2\mathrm{d}x + \|\sigma(m_\beta)-\sigma(\bar{m})x^2\|_\infty\|\mathcal{A}v\|_2^2\end{aligned}$$

By the assumption that $|a(t)| \leq 1$, there is a constant $C(\beta, J)$ such that

$$\|\big(\sigma(m_\beta)-\sigma(\bar{m})\big)x^2\|_\infty \leq C(\beta,J) \tag{3.13}$$

and hence for δ_0 small enough,

$$\|\big(\sigma(m_\beta)-\sigma(\bar{m})\big)x^2\|_\infty\|\mathcal{A}v\|_2^2 \leq 1/2\ .$$

and thus,

$$\sigma(m_\beta)\|x\mathcal{A}v\|_2^2 \leq \int\sigma(\bar{m})x^2\big(\mathcal{A}v\big)^2\mathrm{d}x + \frac{1}{2} \leq \phi(t) \tag{3.14}$$

and the desired result is obtained in this case since $\tilde{\alpha}^2\sigma(m_\beta)^2 = (1-\sigma(m_\beta))^2$.

We emphasize that neither (3.13) nor (3.14) depend on the temporary assumption of antisymmetry for their validity.

To prove the theorem as stated, we drop the assumption that the initial data is antisymmetric, and introduce the smearing operator

$$\mathcal{S}w(x) = \frac{1}{2m_\beta}\bar{m}'\star w(x)\ .$$

Notice that $\mathcal{S}$ is a contraction on L^2, and it commutes with differentiation. Hence,

$$\|\big(\mathcal{A}v\big)'\|_2 \geq \|\mathcal{S}\big(\mathcal{A}v\big)'\|_2 = \|\big(\mathcal{S}\mathcal{A}v\big)'\|_2\ .$$

Now, as explained above, $\mathcal{S}\mathcal{A}v(a(t)) = 0$ and hence the constrained uncertainty principle applies with the result that

$$\|\big(\mathcal{S}\mathcal{A}v\big)'\|_2^2 \geq \frac{9}{4}\frac{\|\mathcal{S}\mathcal{A}v\|_2^4}{\|(x-a(t))\mathcal{S}\mathcal{A}v\|_2^2}\ . \tag{3.15}$$

We now need to remove $\mathcal{S}$. In the numerator, we use Lemma 2.2 as follows: For all $\epsilon > 0$,

$$\begin{aligned}\|\mathcal{S}\mathcal{A}v\|_2^2 =&\|\mathcal{A}v + (\mathcal{S}\mathcal{A}v-\mathcal{A}v)\|_2^2 \geq \\ &(1-\epsilon)\|\mathcal{A}v\|_2^2 - \frac{1}{\epsilon}\|(\mathcal{S}\mathcal{A}v-\mathcal{A}v)\|_2^2 \geq \\ &(1-\epsilon)\|\mathcal{A}v\|_2^2 - \frac{1}{\epsilon}\epsilon_1 C(\beta,J)\big[\mathcal{F}(m(t))-\mathcal{F}(\bar{m})\big]\end{aligned}$$

by Lemma 2.3 and (3.2). Hence,

$$\|\mathcal{SA}v\|_2^2 \geq (1-2\epsilon)\|\mathcal{A}v\|_2^2 \tag{3.16}$$

for ϵ_1 small enough.

To remove $\mathcal{S}$ from the denominator, write

$$\int (x-a(t))^2\big(\mathcal{SA}v(x)\big)^2\mathrm{d}x \leq (1+\epsilon)\int (x)^2\big(\mathcal{SA}v(x)\big)^2\mathrm{d}x + \left(\frac{1+\epsilon}{\epsilon}\right)a(t)^2\|\mathcal{SA}v\|_2^2 \tag{3.17}$$

By Minkowski's inequality and the by–now familiar rule for commuting convolution with multiplication by x, one has

$$\|x\mathcal{SA}v\|_2 \leq \|\mathcal{S}x\mathcal{A}v\|_2 + \|\tilde{\mathcal{S}}\mathcal{A}v\|_2$$

where $\tilde{\mathcal{S}}$ denotes convolution by $(2m_\beta)^{-1}x\bar{m}'(x)$. Clearly $\tilde{\mathcal{S}}$ is bounded on L^2 with norm no greater than $(2m_\beta)^{-1}\|x\bar{m}'(x)\|_1$. And since $\mathcal{S}$ is a contraction on L^2, one has

$$\|x\mathcal{SA}v\|_2 \leq \|x\mathcal{A}v\|_2 + (2m_\beta)^{-1}\|x\bar{m}'(x)\|_1\|\mathcal{A}v\|_2 \ .$$

Thus, for all $\epsilon > 0$,

$$\|x\mathcal{SA}v\|_2^2 \leq (1+\epsilon)\|x\mathcal{A}v\|_2^2 + \left(\frac{1+\epsilon}{\epsilon}\right)(2m_\beta)^{-2}\|x\bar{m}'(x)\|_1^2\|\mathcal{A}v\|_2^2 \ . \tag{3.18}$$

Combining (3.18) and (3.17), and recalling the hypothesis that $|a(t)| < 1$, one has

$$\int (x-a(t))^2\big(\mathcal{SA}v(x)\big)^2\mathrm{d}x \leq (1+\epsilon)^2\|x\mathcal{A}v\|_2^2 + \frac{1}{2} \tag{3.19}$$

when $||v||_2$ is sufficiently small. Combining (3.19) with the first inequality in (3.14), we obtain

$$\int (x-a(t))^2\big(\mathcal{SA}v(x)\big)^2\mathrm{d}x \leq (1+\epsilon)^2\phi(t) \ . \tag{3.20}$$

Then from (3.15), (3.16) and (3.19), we have that

$$\|\big(\mathcal{A}v\big)'\|_2^2 \geq \frac{1-2\epsilon}{(1+\epsilon)^2}\frac{9}{4}\frac{\|\mathcal{A}v\|_2^4}{\phi(t)}$$

for ϵ_1 and δ_0 small enough, without any extra assumptions on v, and then from (3.11), one obtains the final result. ■

4 Moment Estimates

Our goal in this section is to estimate the growth of

$$\phi(t) = \int \sigma(\bar{m})|x\mathcal{A}v|^2 \mathrm{d}x + 1 \tag{4.1}$$

where v is related to a solution m of (1.1) through (1.21), under the hypotheses that

$$\mathcal{I}(m(t)) << \left[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})\right] \tag{4.2}$$

and that

$$|a(t)| \leq 1 \ . \tag{4.3}$$

Recall that the notational convention, (1.22), is that $\bar{m}$ stands for $\bar{m}_{a(t)}$ and that $\mathcal{A}$ denotes the second variation of $\mathcal{F}$ at $\bar{m}_{a(t)}$. The main result of this section is the following:

Theorem 4.1 *Let $m(\cdot, t)$ be a solution of (1.1). Then for any $\epsilon > 0$ there are constants $\kappa_0(\beta, J, \epsilon)$, $\delta_0(\beta, J, \epsilon)$ and $\epsilon_1(\beta, J, \epsilon)$ depending only on β, J and ϵ, such that for all t with*

$$\mathcal{I}(m(t)) \leq \epsilon_1 \left[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})\right] \tag{4.4}$$

$||v'(t)||_2 < \kappa_0$, $||v(t)||_2 < \delta_0$ and $|a(t)| \leq 1$, it is the case that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \leq (1+\epsilon)4(1-\sigma(m_\beta))^2 \left[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})\right] \ . \tag{4.5}$$

Our starting point is the following result, Lemma 4.5 of [3], that we reproduce below:

Lemma 4.2 *Let $v(t)$ be related to a solution $m(t)$ of (1.1) through (1.21). Then for any $\epsilon > 0$, there are constants $\delta(\beta, J, \epsilon) > 0$ and $\kappa(\beta, J, \epsilon) > 0$ such that for all t with $\|v(t)\|_2 \leq \delta(\beta, J, \epsilon)$ and $\|v'(t)\|_2 \leq \kappa(\beta, J, \epsilon)$*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \leq \\ &-4\int \mathcal{A}\big(\sigma(\bar{m})\mathcal{A}v\big)\big(\sigma(\bar{m})x(\mathcal{A}v)'\big)\mathrm{d}x - 2\|\mathcal{A}^{1/2}\big(\sigma(\bar{m})x\big(\mathcal{A}v\big)'\big)\|_2^2 + \\ &I_1 + I_2 + I_3 + I_4 + \\ &\epsilon\big[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})\big] + \epsilon\|\mathcal{A}^{1/2}\big(\sigma(\bar{m})x\big(\mathcal{A}v\big)'\big)\|_2^2 \end{aligned} \tag{4.6}$$

where

$$I_1 = -2\int g\big(\sigma(\bar{m})x^2\mathcal{A}v\big)\big(\sigma(\bar{m})(\mathcal{A}v)'\big)\mathrm{d}x$$

$$I_2 = -2\int \mathcal{A}\big(\sigma(\bar{m})'x^2\mathcal{A}v\big)\big(\sigma(\bar{m})(\mathcal{A}v)'\big)\mathrm{d}x$$

$$I_3 = -4\int \mathcal{C}\big(\sigma(\bar{m})\mathcal{A}v\big)\big(\sigma(\bar{m})(\mathcal{A}v)'\big)\mathrm{d}x$$

$$I_4 = -2\int \mathcal{C}\big(\sigma(\bar{m})x\big(\mathcal{A}v\big)'\big)\big(\sigma(\bar{m})(\mathcal{A}v)'\big)\mathrm{d}x$$

and where

$$g(x) = \frac{2\bar{m}\bar{m}'}{(1-\bar{m}^2)^2} \tag{4.7}$$

and for any function $w(x)$*,*

$$\mathcal{C}w(x) = \int J(y)yw(x-y)\mathrm{d}y \ . \tag{4.8}$$

Proof of Theorem 4.1: We begin by estimating I_1 through I_4.

$$\begin{aligned} I_1 = -2\langle \big(g\sigma(\bar{m})^2x^2\big)\mathcal{A}v, \big(\mathcal{A}v\big)'\rangle_{L^2} \le \\ 2\|g\sigma(\bar{m})^2x^2\|_\infty \|\mathcal{A}v\|_2 \|\big(\mathcal{A}v\big)'\|_2 \le \\ 2\|g\sigma(\bar{m})^2x^2\|_\infty \left(\frac{2\tilde{\alpha}\epsilon_1(1+2\epsilon)}{(1-\epsilon)\sigma(m_\beta)}\right)^{1/2} \big[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})\big] \end{aligned} \tag{4.9}$$

where we used (3.10) and (3.12) in the last step. Note that $\|g\sigma(\bar{m})^2x^2\|_\infty$ is bounded by a constant depending only on β and J by (1.8) and the hypothesis that $|a(t)| \le 1$, since (1.8) implies that g is a rapidly decaying bump function centered on $a(t)$. Other L^∞ estimates involving x will be treated in the same way without further mention. This is the only use made of $|a(t)| \le 1$.

Similarly,

$$\begin{aligned} I_2 = -2\langle \sigma(\bar{m})\mathcal{A}\big(\sigma(\bar{m})'x^2\mathcal{A}v\big), \big(\mathcal{A}v\big)'\rangle_{L^2} \le \\ 2\|\sigma(\bar{m})\|_\infty \|\|\mathcal{A}\|\| \|\sigma(\bar{m})'x^2\|_\infty \|\mathcal{A}v\|_2 \|\big(\mathcal{A}v\big)'\|_2 \le \\ 2\|\sigma(\bar{m})\|_\infty \|\|\mathcal{A}\|\| \|\sigma(\bar{m})'x^2\|_\infty \left(\frac{2\tilde{\alpha}\epsilon_1(1+2\epsilon)}{(1-\epsilon)\sigma(m_\beta)}\right)^{1/2} \big[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})\big] \end{aligned} \tag{4.10}$$

In the same way, one obtains similar bounds for I_3 and I_4 and then since all of the $\|\cdot\|_\infty$ terms are bounded *a–priori* in terms of β and J, there is a constant $C(\beta, J)$ depending only on β and J such that

$$I_1 + I_2 + I_3 + I_4 \le C(\beta, J)\epsilon_1 \big[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})\big] \ .$$

Choosing $\epsilon_1 = \epsilon / C(\beta, J)$, one has

$$I_1 + I_2 + I_3 + I_4 \le \epsilon \big[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})\big] \ . \tag{4.11}$$

The following notational conventions allow the rest of the proof to be expressed more clearly. Let

$$f = \sigma(\bar{m})\mathcal{A}v \tag{4.12}$$

and

$$h = \sigma(\bar{m})x(\mathcal{A}v)' \ . \tag{4.13}$$

Then from (4.11) and (4.6) one has

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \le -4\langle f, \mathcal{A}h\rangle_{L^2} - (2-\epsilon)\langle h, \mathcal{A}h\rangle_{L^2} + \\ 2\epsilon\big[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})\big] \end{aligned} \tag{4.14}$$

Now, since $\mathcal{A}$ is non negative,

$$
\begin{aligned}
&-4\langle f, \mathcal{A}h\rangle_{L^2} - (2-\epsilon)\langle h, \mathcal{A}h\rangle_{L^2} = \\
&\quad -\langle (2-\epsilon)^{1/2}h + 2(2-\epsilon)^{-1/2}f, \mathcal{A}\big((2-\epsilon)^{1/2}h + 2(2-\epsilon)^{-1/2}f\big)\rangle_{L^2} + 4(2-\epsilon)^{-1}\langle f, \mathcal{A}f\rangle_{L^2} \le \\
&\quad\quad 4(2-\epsilon)^{-1}\langle f, \mathcal{A}f\rangle_{L^2}
\end{aligned}
$$

To bound this in terms of the excess free energy, one makes repeated use of (2.13) of Lemma 2.3, together with the self adjointness and boundedness of $\mathcal{A}$, to replace factors of $\sigma(\bar{m})$ with factors of $\sigma(m_b)$:

$$
\begin{aligned}
&\langle f, \mathcal{A}f\rangle_{L^2} = \\
&\langle \sigma(\bar{m})\mathcal{A}v, \mathcal{A}\sigma(\bar{m})\mathcal{A}v\rangle_{L^2} \le \\
&\sigma(m_\beta)\langle \mathcal{A}v, \mathcal{A}\sigma(\bar{m})\mathcal{A}v\rangle_{L^2} + C\|\big(\mathcal{A}v\big)'\|_2\|\mathcal{A}v\|_2 = \\
&\sigma(m_\beta)\langle \mathcal{A}^2v, \sigma(\bar{m})\mathcal{A}v\rangle_{L^2} + C\|\big(\mathcal{A}v\big)'\|_2\|\mathcal{A}v\|_2 \le \\
&\sigma(m_\beta)^2\langle \mathcal{A}^2v, \sigma(\bar{m})\mathcal{A}v\rangle_{L^2} + 2C\|\big(\mathcal{A}v\big)'\|_2\|\mathcal{A}v\|_2 \le \\
&\sigma(m_\beta)^2\tilde{\alpha}\langle \mathcal{A}v, \mathcal{A}v\rangle_{L^2} + 3C\|\big(\mathcal{A}v\big)'\|_2\|\mathcal{A}v\|_2 = \\
&\sigma(m_\beta)^2\tilde{\alpha}\langle v, \mathcal{A}^2v\rangle_{L^2} + 4C\|\big(\mathcal{A}v\big)'\|_2\|\mathcal{A}v\|_2
\end{aligned}
$$

where C is constant derived from those in the cited lemmas. By Lemma A.2 of [3], $\|\mathcal{A}v\|_2^2 \le C\big[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})\big]$ for δ and κ sufficiently small, and hence by the hypotheses (4.4), we have

$$
4C\|\big(\mathcal{A}v\big)'\|_2\|\mathcal{A}v\|_2 \le \epsilon\big[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})\big]
$$

for ϵ_1 sufficiently small.

Combining estimates and redefining ϵ, one has

$$
\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \le (1+\epsilon)4\tilde{\alpha}^2\sigma(m_\beta)^2\big[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})\big] \ . \tag{4.15}
$$

which is the desired result since $\tilde{\alpha}^2\sigma(m_\beta)^2 = (1-\sigma(m_\beta))^2$.

5 Proof of the Main Results

We begin this section by proving several lemmas concerning the L^1 norm. The first of these will be used in the proof of Theorem 1.1 to control $|a - a(t)|$ when a is given by (1.11) and $a(t)$ is the center of the closest front in L^2, as in (1.19).

Lemma 5.1 *Let w be any integrable function, and let the variable b be fixed by the condition that*

$$\int \big(w(x) - \bar{m}_b(x)\big)\mathrm{d}x = 0 \ . \tag{5.1}$$

Then for any c

$$|b - c| \le \frac{1}{2m_\beta}\int \big|w(x) - \bar{m}_c(x)\big|\mathrm{d}x \ . \tag{5.2}$$

In particular, for any solution $m(t)$ of (1.1), and any t such that $m(t)$ is integrable,

$$|a(t) - a| \le \frac{1}{2m_\beta}\int \big|m(x,t) - \bar{m}_{a(t)}(x)\big|\mathrm{d}x \ . \tag{5.3}$$

where a is fixed by the condition that

$$\int \big(m(x,t) - \bar{m}_a(x)\big)\mathrm{d}x = 0 \ . \tag{5.4}$$

Proof: First, since $\int \bar{m}_0'(x)\mathrm{d}x = 2m_\beta > 0$, there is exactly one a such that (5.1) holds. Next, adding and subtracting $\bar{m}_b$, one sees

$$\int \big(w(x) - \bar{m}_b(x)\big)\mathrm{d}x = \int \big(\bar{m}_a(x) - \bar{m}_b(x)\big)\mathrm{d}x$$

Also, it is clear that

$$\int \big(\bar{m}_a(x) - \bar{m}_b(x)\big)\mathrm{d}x = 2m_\beta(b - a)$$

and (5.2) easily follows. ■

Lemma 5.2 *Let w be any function such that*

$$\int |w(x)|^2(1 + x^2)\mathrm{d}x < \infty \ . \tag{5.5}$$

Then for any $0 < \delta < 1$,

$$\|w\|_1 \le C(\delta)\|(1 + x^2)^{1/2}w\|_2^{(1+\delta)/2}\|w\|_2^{(1-\delta)/2}$$

where

$$C(\delta) = \left(\int (1 + x^2)^{-(1+\delta)/2}\mathrm{d}x\right)^{1/2}$$

which is finite.

Proof: Let $p = (1+\delta)/2$, and observe that

$$
\begin{aligned}
\int |w(x)|\mathrm{d}x = \int (1+x^2)^{-p/2}(1+x^2)^{p/2}|w(x)|\mathrm{d}x \leq \\
\left(\int (1+x^2)^{-p}\mathrm{d}x\right)^{1/2}\left(\int (1+x^2)^{p}|w(x)|^2\mathrm{d}x\right)^{1/2}
\end{aligned}
\tag{5.6}
$$

Next, since $p < 1$, Jensen's inequality implies

$$
\frac{1}{\|w\|_2^2}\int (1+x^2)^p|w(x)|^2\mathrm{d}x \leq \left(\frac{1}{\|w\|_2^2}\int (1+x^2)|w(x)|^2\mathrm{d}x\right)^p \ . \tag{5.7}
$$

The result easily follows from (5.6) and (5.7). ■

Since we work in the proof with moments of $\mathcal{A}v$ instead of v, we need one more lemma to apply the previous one.

Lemma 5.3 *There is a finite constant $D(\beta, J)$ depending only on β and J so that for all t such that $|a(t)| \leq 1$ and $\|v(t)\|_2 \leq 1$,*

$$
\|(1+x^2)^{1/2}v(t)\|_2^2 \leq D(\beta, J)\phi(t) \tag{5.8}
$$

and

$$
\big(\phi(t) - 1\big) \leq D(\beta, J)\|xv(t)\|_2^2 \ . \tag{5.9}
$$

Proof: Let $\mathcal{B}$ be the operator defined by

$$
\mathcal{B}w = \frac{1}{\beta(1-m_\beta^2)}w - J \star w \ .
$$

As we have pointed out in [3], this operator is bounded and has a bounded inverse on L^2. Then, using once more the by–now familiar rules for comuting convolution and multiplication by x,

$$
\begin{aligned}
\|xv\|_2 \leq & \|\mathcal{B}^{-1}\|\|\mathcal{B}xv\|_2 = \\
& \|\mathcal{B}^{-1}\|\|x\mathcal{B}v - (xJ)\star v\|_2 \leq \\
& \|\mathcal{B}^{-1}\|\big(\|x\mathcal{B}v\|_2 + \|(xJ)\|_1\|v\|_2\big) \leq \\
& D\big(\|x\mathcal{B}v\|_2 + \|v\|_2\big)
\end{aligned}
\tag{5.10}
$$

for some constant D depending only on β and J. Next,

$$
\begin{aligned}
\|x\mathcal{B}v\|_2 = & \|x\mathcal{A}v + x\tilde{g}v\|_2 \leq \\
& \|x\mathcal{A}v\|_2 + \|x\tilde{g}\|_\infty\|v\|_2 \leq \\
& D\big(\|x\mathcal{A}v\|_2 + \|v\|_2\big)
\end{aligned}
\tag{5.11}
$$

where

$$
\tilde{g}(x) = \sigma(m_\beta)^{-1} - \sigma(\bar{m})^{-1} \ ,
$$

and D is some other finite constant depending only on β and J, since y (1.8) and the hypothesis $|a(t)| \leq 1$, $\|x\tilde{g}\|_\infty \leq C(\beta,J)$ for some constant $C(\beta,J)$ depending only on β and J. Finally,

$$\begin{aligned}\sigma(m_\beta)\|x\mathcal{A}v\|_2^2 = &\int x^2(\sigma(m_\beta)-\sigma(\bar{m}))\big(\mathcal{A}v\big)^2\mathrm{d}x + \int x^2\sigma(\bar{m})\big(\mathcal{A}v\big)^2\mathrm{d}x \leq \\ &\|x^2(\sigma(m_\beta)-\sigma(\bar{m}))\|_\infty\|\mathcal{A}v\|_2^2 + \int x^2\sigma(\bar{m})\big(\mathcal{A}v\big)^2\mathrm{d}x\end{aligned} \tag{5.12}$$

and the sup norm is again bounded by some constant $C(\beta,J)$ depending only on β and J by (1.8) and the hypothesis $|a(t)| \leq 1$.

Combining these estimates, one easily obtains

$$\|(1+x^2)v\|_2^2 \leq D(\beta,J)\left(\int x^2\sigma(\bar{m})\big(\mathcal{A}v\big)^2\mathrm{d}x + \|v\|_2^2\right)$$

which yields (5.8) since $\|v(t)\|_2 \leq 1$ by hypothesis. The prrof of (5.9) simply reverses the above steps. With D changing from line to line, one easily obtains

$$\begin{aligned}\int x^2\sigma(\bar{m})\big(\mathcal{A}v\big)^2\mathrm{d}x \leq &D\big(\|x\mathcal{A}v\|_2^2 + \|v\|_2^2\big) \leq \\ &D\big(\|x\mathcal{B}v\|_2^2 + \|v\|_2^2\big) \leq \\ &D\big(\|\mathcal{B}xv\|_2^2 + \|v\|_2^2\big) \leq \\ &D\big(\|xv\|_2^2 + \|v\|_2^2\big) \leq\end{aligned}$$

■

Proof of Theorem 1.1 : First, fix $\epsilon > 0$, and then choose δ_1, κ_1 and ϵ_1 small enough so that both the following three estimates hold under the condition that

$$\mathcal{I}(m(t)) \leq \epsilon_1\big[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})\big] \tag{5.13}$$

for all t such that $|a(t)| < 1$, $\|v(t)\|_2 \leq \delta_1$ and $\|v'(t)\|_2 \leq \kappa_1$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar{m}\big)\big] \leq -9(1-\epsilon)(1-\sigma(m_\beta))^2\frac{\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar{m}\big)\big]^2}{\phi(t)} \tag{5.14}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \leq (1+\epsilon)4(1-\sigma(m_\beta))^2\big[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})\big]\ . \tag{5.15}$$

This is possible by Theorems 3.1 and 4.1 of the present paper. By Theorem 4.1 and Lemma 1.2 of [3], decreasing $\delta_1 > 0$ and $\kappa_1 > 0$ if need be, we have finite constants B and C such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \leq B\big[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})\big] \tag{5.16}$$

and

$$\frac{1}{C}\|v(t)\|_2^2 \leq \big[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})\big] \leq C\|v\|_2^2 \tag{5.17}$$

for all t such that $|a(t)| < 1$, $\|v(t)\|_2 \leq \delta_1$ and $\|v'(t)\|_2 \leq \kappa_1$.

Next define δ_0 by

$$\delta_0 = \frac{\delta_1}{4(C^2+1)} \tag{5.18}$$

where C is the constant in (5.17). Theorem 2.2 of [3], applied with the values of δ_0, δ_1 and κ_1 fixed above, guarantees the existence of an $\epsilon_0 > 0$ and a t_0 so that when the initial data satisfies $\|m_0 - \bar{m}\|_2 \leq \epsilon_1$, the solution to (1.1) satisfies

$$\|v(t_0)\|_2 \leq \delta_0 \tag{5.19}$$

and

$$\|v'(t)\|_2 \leq \kappa_1$$

for all $t \geq t_0$ such that $\|v(t)\|_2 \leq \delta_1$.

Next, we have from Theorem 2.2 of [3] that

$$\int \big(m(x,t_0) - \bar{m}_0(x)\big)^2 \mathrm{d}x \leq 2c_0$$

where c_0 is the constant specified in the hypotheses of Theorem 1.1. Clearly then,

$$\|xv(t_0)\|_2^2 \leq 2\big(\|x(m(t_0) - \bar{m}_0\|_2^2 + 4m_\beta a(t_0)\big) \ .$$

By Lemma 2.4 of [3] we may suppose, further decreasing δ_1 if need be, that $4m_\beta a(t_0) \leq c_0$. Then

$$\|xv(t_0)\|_2^2 \leq 5c_0$$

and hence, by (5.9) of Lemma 5.3,

$$\phi(t_0) \leq \tilde{c}_0 \tag{5.20}$$

where $\tilde{c}_0$ is a finite constant depending only on c_0, β and J. Hence, we have control on the values of both $f(t_0)$ and $\phi(t_0)$ through (5.19) and (5.20).

The time t_0 is the time we have to wait for the smoothing properties of the equation (1.1) to regularize our data enough that the estimates above all hold, and it fixes the left end of the interval on which we shall work. To fix the right end, which we shall eventually show to be $+\infty$, define

$$T_0 = \min\{\ \inf\{\ t > t_0 \mid \|v(t)\|_2 \geq \delta_1/2\ \}\ ,\ \inf\{\ t > t_0 \mid |a(t)| \geq 1\ \}\ \}\ .$$

Then, uniformly on the interval (t_0, T_0), both of the estimates (5.16) and (5.17) holds. Moreover for those t in (t_0, T_0) such that (5.13) holds, one also has (5.14) and (5.15). Hence, writing $f(t) = \big[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})\big]$, we have the following alternative:

One the one hand, in case

$$\mathcal{I}(m(t)) \leq \epsilon_1 \big[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})\big] \tag{5.21}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f(t) &\leq -\tilde{A}\frac{f(t)^2}{\phi(t)} \\ \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) &\leq \tilde{B} f(t) \end{aligned} \tag{5.22}$$

where $\tilde{A}$ and $\tilde{B}$ by

$$\begin{aligned} \tilde{A} &= 9(1-\epsilon)(1-\sigma(m_\beta))^2 \\ \tilde{B} &= 4(1+\epsilon)(1-\sigma(m_\beta))^2 \end{aligned} \tag{5.23}$$

On the other hand, in case

$$\mathcal{I}(m(t)) \geq \frac{\epsilon_1}{2}\big[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})\big] \tag{5.24}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f(t) &\leq -\frac{\epsilon_1}{2} f(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) &\leq B f(t) \end{aligned} \tag{5.25}$$

In the application of the system of differential inequalities (1.32), it is the *ratio* of the constants A and B that determines the exponent q. Indeed,

$$q = \frac{(A/B)}{(A/B)+1}$$

The values of A and B themselves can be changed, keeping this ratio fixed, simply by rescaling the time t. Therefore we define

$$A = \frac{\tilde{A}}{\tilde{B}} B$$

and observe that

$$\frac{\epsilon_1}{2} f(t) = \frac{\epsilon_1}{2Af(t)} A f(t)^2 \geq \frac{\epsilon_1}{2Af(t)} A \frac{f(t)^2}{\phi}$$

since $\phi(t) \geq 1$ by definition. Now, by (5.17) we may further decrease $\delta_1 > 0$ if need be to ensure that

$$\frac{\epsilon_1}{2Af(t)} \geq 1 \ .$$

Doing so, we have that in case

$$\mathcal{I}(m(t)) \geq \frac{\epsilon_1}{2}\big[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})\big]$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f(t) &\leq -A\frac{f(t)^2}{\phi(t)} \\ \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) &\leq B f(t) \end{aligned} \tag{5.26}$$

where

$$\frac{A}{B} = \frac{\tilde{A}}{\tilde{B}} \ .$$

Now suppose that at t_0,

$$\mathcal{I}(m(t_0)) > \frac{\epsilon_1}{2}\big[\mathcal{F}(m(t_0)) - \mathcal{F}(\bar{m})\big] \ .$$

Define

$$t_1 = \inf\{\ t > t_0 \mid \mathcal{I}(m(t)) \leq \frac{\epsilon_1}{2}\big[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})\big]\} \ .$$

Note that by Theorem 2.2 of [3],

$$\frac{\mathcal{I}(m(t))}{\big[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})\big]} \tag{5.27}$$

is continuous and even differentiable. Hence $t_1 > t_0$. Next, define

$$t_2 = \inf\{\ t > t_1 \mid \mathcal{I}(m(t)) \geq \epsilon_1\big[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})\big]\} \ ,$$

$$t_3 = \inf\{\ t > t_2 \mid \mathcal{I}(m(t)) \leq \frac{\epsilon_1}{2}\big[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})\big]\} \ ,$$

and so forth. (We follow the usual convention that if there is no $t < T_0$ satisfying the condition, the infimum is set to be T_0.) The sequence of times t_j can have no limit point except possibly T_0, since at such a limit point, the continuous function (5.27) would take on two values.

If at t_0,

$$\mathcal{I}(m(t_0)) \le \frac{\epsilon_1}{2}\big[\mathcal{F}(m(t_0)) - \mathcal{F}(\bar{m})\big] \ ,$$

one would define

$$t_1 = \inf\{\ t > t_0 \mid \mathcal{I}(m(t)) \ge \epsilon_1\big[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})\big]\} \ ,$$

and then proceed as above with the opposite alternation.

In either case, one produces a sequence of intervals $[t_j, t_{j+1}]$ on which (5.22) and (5.26) hold in successive alternation. On each of these intervals, we may apply Theorem 5.1 of [3]. To put all of these estimates together in a transparent way, we rescale the intervals on which (5.26) holds. Supposing that (5.26) holds on $[t_0, t_1]$, define

$$s(t) = \frac{A}{\tilde{A}}(t - t_0) \qquad \text{and} \qquad s_1 = \frac{A}{\tilde{A}}(t_1 - t_0)$$

for $t_0 < t < t_1$,

$$s(t) = s_1 + (t - t_1) \qquad \text{and} \qquad s_2 = s_1 + (t_2 - t_1)$$

for $t_1 < t < t_2$,

$$s(t) = s_2 + \frac{A}{\tilde{A}}(t - t_2) \qquad \text{and} \qquad s_3 = \frac{A}{\tilde{A}}(t_3 - t_2)$$

for $t_2 < t < t_3$, and so forth in alternation. It follows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}f(s) &\le -\tilde{A}\frac{f(s)^2}{\phi(s)} \\ \frac{\mathrm{d}}{\mathrm{d}s}\phi(s) &\le \tilde{B}f(s) \end{aligned} \tag{5.28}$$

for all s with $0 \le s \le s(T_0)$. By Theorem 5.1 of [3],

$$\begin{aligned} f(s) &\le f(0)^{1-q}\phi(0)^q\left(\frac{\phi(0)}{f(0)} + (\tilde{A}+\tilde{B})s\right)^{-q} \\ \phi(s) &\le f(0)^{1-q}\phi(0)^q\left(\frac{\phi(0)}{f(0)} + (\tilde{A}+\tilde{B})s\right)^{1-q} \end{aligned} \tag{5.29}$$

where

$$q = \frac{\tilde{A}}{\tilde{A}+\tilde{B}}$$

and where $f(0)$ and $\phi(0)$ are bounded by (5.19) and (5.20). Now, for any $\delta > 0$, we can choose ϵ so that

$$\frac{\tilde{A}}{\tilde{A}+\tilde{B}} = \frac{9}{13} - \delta \ . \tag{5.30}$$

We shall now show that for δ small enough, $|a(t)| \le 1/2$ for all $t \le T_0$. Then by Lemmas 5.2, 5.3 and (5.17),

$$\begin{aligned} \|v(s)\|_1 \le & C(\delta)\|(1+x^2)^{1/2}v(s)\|_2^{(1+\delta)/2}\|v(s)\|_2^{(1-\delta)/2} \le \\ & C(\delta)D(\beta,J)^{(1+\delta)/4}C^{(1-\delta)/4}\phi(s)^{(1+\delta)/4}f(s)^{(1-\delta)/4} \le \\ & C(\delta)D(\beta,J)^{(1+\delta)/4}C^{(1-\delta)/4}f(0)^{(1-q)/4)}\phi(0)^{q/4}\left(\frac{\phi(0)}{f(0)} + (\tilde{A}+\tilde{B})s\right)^{(1-2q+\delta)/4} \end{aligned} \tag{5.31}$$

The right hand side is decreasing for $\delta < 5/26$, and we now choose δ to be at least this small. Moreover, the value at $s = 0$ can be made arbitrarily small by decreasing δ_1. We now do so, if need be, to ensure that

$$\|v(s)\|_1 \le m_\beta/2$$

for all $s \le s(T_0)$. Hence, by Lemma 5.1,

$$|a(t) - a| \le 1/4$$

for all $t \le T_0$. But then by Lemma 5.1 again, this implies that $|a(t)| < 1/2$ for all $t \le T_0$.

Hence if $T_0 < \infty$, it is because $\|v(T_0)\|_2 = \delta_1/2$. But since (5.17) is still valid with the same constant C on the closed interval $[t_0, T_0]$, and since the excess free energy is monotone decreasing,

$$\begin{aligned}&\frac{\delta_1^2}{4} = \|v(T_0)\|_2^2 \le \\ &C\big(\mathcal{F}(\bar m + v(T_0)) - \mathcal{F}(\bar m)\big) \le C\big(\mathcal{F}(\bar m + v(t_0)) - \mathcal{F}(\bar m)\big) \le \\ &C^2\|v(t_0)\|_2^2 \le C^2\delta_1^2\end{aligned}$$

This contradicts (5.18), and hence $T_0 < \infty$ is not possible.

We now clearly have (1.16) since

$$s(t) \ge \min\left\{\frac{A}{\tilde A}\ ,\ 1\right\}(t - t_0)\ .$$

Also from this and (5.31), we have

$$\|m(t) - \bar m_{a(t)}\|_1 \le c_2(1 + c_1 t)^{-(5/52-\delta)}\ .$$

But

$$\begin{aligned}&\|m(t) - \bar m_a\|_1 \le \\ &\|m(t) - \bar m_{a(t)}\|_1 + \|\bar m_a - \bar m_{a(t)}\|_1 = \\ &\|m(t) - \bar m_{a(t)}\|_1 + 2m_\beta|a - a(t)| \le \\ &(1 + 2m_\beta)\|m(t) - \bar m_{a(t)}\|_1\end{aligned}$$

and hence (1.17) follows as well. ■

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