Abstract. We consider multiscale functions of the type that are studied in averaging and homogenization theory and in multiscale modeling. Typical examples are two-scale functions $f(x,x/\epsilon)$, $(0 < \epsilon << 1)$ that are periodic in the second variable. We prove that under certain band limiting conditions these multiscale functions can be uniquely and stably recovered from non-uniform samples of optimal rate. The goal of this study is to establish the close relation between computational grids in multiscale modeling and sampling strategies developed in information theory.

Key words. Shannon's sampling theorem, nonuniform periodic sampling, multiscale functions, heterogeneous multiscale method

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1. Introduction. Multiscale modeling and computation has recently been a very active research field. A major computational challenge is that in direct numerical simulation the smallest important scales must be resolved over the length of the largest scales in each dimension. This can lead to a prohibitively high computational cost. A number of different numerical frameworks have been proposed to handle this problem and we will here focus on the heterogeneous multiscale method (HMM) [1, 6]. The purpose of this work is to study multiscale modeling from the point of view of information theory.

Even the basic cost of direct numerical simulation can be understood from information theory. The classical Shannon sampling theorem gives that just in order to represent the solution, at least two unknowns per wavelength is required [15]. If the size of the computational domain is 1 and $\epsilon << 1$ is the smallest important wavelength, then at least $2\epsilon^{-1}$ unknowns are required in each dimension. The computational complexity in $d$ dimensions must then be at least $O(\epsilon^{-d})$.

If this is too much for the available computational resources then some special features of the original problem must be exploited. Scale separation is assumed in homogenization theory, [3], and in convergence analysis of HMM. The functions involved are typically of the form,

\begin{equation}
(1.1) \quad f'(x) = f(x,x/\epsilon) \quad \text{where } f(x,y) \text{ is } \Gamma\text{-periodic in } y, \quad 0 < \epsilon << 1
\end{equation}

where $\Gamma = [0,1]^d$.

With equidistant sampling points the rate must still be the same as above $O(\epsilon^{-d})$ in order to recover the function. Different sampling strategies are required to exploit the special structure of the functions. This corresponds to strategies for numerical computational grids in multiscale simulations. We will see that some techniques of multiscale computations actually are optimal if seen via information theory.

For $g \in L_2(\mathbb{R}^d)$ we denote the Fourier transform $\hat{g} \in L_2(\mathbb{R}^d)$ by

$$\hat{g}(\xi) = \int_{\mathbb{R}^d} g(x)e^{2\pi i x \cdot \xi} dx,$$
where the Euclidean dot product is \( \xi \cdot x = \sum_{j=1}^{d} \xi_j x_j \). Functions whose Fourier transform have bounded support are called \textit{bandlimited functions}.

\[
\begin{align*}
\xi & \cdot x = \sum_{j=1}^{d} \xi_j x_j \\
\text{Functions whose Fourier transform have bounded support are called} & \text{ \textit{bandlimited functions}.}
\end{align*}
\]

The class of multiscale functions \( f^\epsilon(x) \) defined by (1.1) possess scale separation between the \( O(1) \) “slow” oscillations and \( O(\epsilon^{-1}) \) “fast” oscillations when \( f(x,y) \) is a bandlimited function in both variables. In §3 we will provide derivations of the properties summarized below.

We will start by assuming that the two dimensional Fourier transform of \( f(x,y) \in L_2(\mathbb{R}^2) \) satisfies

\[
(1.2) \quad \text{supp} \left( \hat{f} \right) \subset \bigcup_{m=-M}^{M} (-N,N) \times \{m\}, \quad 0 < 2N < \frac{1}{\epsilon}.
\]

The spectrum of a function \( f^\epsilon \in L_2(\mathbb{R}) \) defined by (1.1 - 1.2) is supported on a finite union of intervals

\[
(1.3) \quad \text{supp} \left( \hat{f}^\epsilon \right) \subset \bigcup_{m=-M}^{M} \left( -N + \frac{m}{\epsilon}, N + \frac{m}{\epsilon} \right).
\]

This representation combines the two notions of “scale” used in information theory and in multiscale computation. The explicit knowledge of the locations of the frequency bands of these functions allows us to design a sampling strategy to reconstruct \( f^\epsilon \) using an optimal sampling rate. This strategy is described in the following theorem.

\textbf{Theorem 1.1.} Suppose the functions \( f^\epsilon \in L_2(\mathbb{R}), f \in L_2(\mathbb{R}^2) \) satisfy the assumptions (1.1 - 1.2), and define the shifted uniform sampling sets \( X_k \) by

\[
(1.4) \quad X_k = \{ j \Delta x + k \delta x \mid j \in \mathbb{Z} \}, \quad -K \leq k \leq K,
\]

where \( 0 < \Delta x < \frac{1}{2N} \) and \( 0 < \delta x < \frac{\epsilon}{2M+1} \). For \( K = M \), the function \( f^\epsilon \) can be uniquely reconstructed from the samples \( f^\epsilon(z), z \in \cup_k X_k \).
Moreover, there is a positive constant \( C = C(\delta x/\epsilon) \) such that the following stability estimate holds

\[
\|f^r\|_2^2 \leq C \sum_{y \in \bigcup_k X_k} |f^r(y)|^2.
\]

(1.5)

The sampling strategy in the theorem above, illustrated in Figure 1.1, is very well matched with the grids used in multiscale methods. As an example, HMM provides a framework for capturing large scale features on coarse grids by incorporating local simulations on grids with much finer resolution, see [1, 6] and §2.2.

In the next section, we give a brief background to relevant issues in multiscale computation and information theory. In §3 we describe the class of bandlimited functions that satisfy (1.1 − 1.2). In §4 we prove a lemma on reconstruction from particular sampling sets that is needed in the proof of the main result. This proof is given in §5 and in §6 the problem with a localized microscale is discussed.

2. Multiscale modeling and sampling strategies. The goal of this paper is to make a connection between computational grid practice in multiscale modeling and information theory and to formulate a sampling strategies for a class of continuous signals that includes cases where the sampling rate may result in aliasing.

2.1. Representation of multiscale functions. Fourier analysis is a standard way of representing signals \( g \in L_2(\mathbb{R}^d) \) in terms of components on different scales. This representation extends to more general decompositions of signals in terms of a sequence \( \{\phi_n\}_{n=-\infty}^{\infty} \subset L_2(\mathbb{R}^d) \)

\[
g(x) = \sum_{n=-\infty}^{\infty} \langle g, \phi_n \rangle \phi_n(x)
\]

(2.1)

where \( \langle \cdot, \cdot \rangle \) defines an inner product on the space. The vectors \( \{\phi_n\} \) can form an orthonormal basis of, for example, trigonometric functions or wavelets.

Here we connect the two: starting with a bandlimited function \( f(x, y) \) that has Fourier decomposition (2.1) given by

\[
f(x, y) = \sum_{n=-\infty}^{\infty} f_n(x) \phi_n(y),
\]

we include a “fast” variable through the transformation \( y \rightarrow x/\epsilon \). The resulting functions \( f^r \) given by (1.1) have a multiscale representation in the viewpoints of both Fourier analysis and periodic homogenization.

2.2. Multiscale computation. We will relate results in information theory to the numerical analysis of a linear or nonlinear system that is discretized on a uniform grid \( X = \{x_j \mid 0 \leq j \leq N\} \) with spacing \( \Delta x = x_j - x_{j-1} \). In order to ensure that the solution to the discretized problem, \( \{u_j\}_{j=0}^{N} \), is consistent with the solution to the true problem and the approximation is stable, the grid \( X \) must be sufficiently dense.

In order to approximate \( u^r \), the solution of a multi scale system, the grid spacing \( \Delta x \) is chosen to be much smaller than the smallest scale in order to fully resolve the \( \epsilon \)-scale. Multiscale algorithms exist that achieve a close approximation to \( u^r \) on much coarser grids. They do this by exploiting the special properties of \( u^r \) such as periodicity, scale separation, and bounded spectral support.
For example, in [1, 8] HMM is used for the solution of stiff ordinary differential equations (ODEs) of the form
\[
\begin{align*}
\frac{du^\epsilon}{dt} &= f(u^\epsilon, v^\epsilon, t) \\
\frac{dv^\epsilon}{dt} &= \frac{1}{\epsilon} g(u^\epsilon, v^\epsilon, t)
\end{align*}
\]
where \(v^\epsilon\) is a solution that oscillates on the time scale of \(O(\epsilon), \epsilon << 1\), and \(u^\epsilon\) mainly varies on the time scale \(O(1)\).

Assume that as \(\epsilon \to 0\), \(u^\epsilon \to U \in C^1(\mathbb{R})\) and that \(U\) is given by
\[
\frac{d}{dt} U = \bar{f}(U, t).
\]
This “effective” system can be solved using HMM even if the form of \(\bar{f}\) is not explicitly known. The right hand side of (2.3) can be approximated using averaged solutions to the full system.

Figure 2.1 represents a HMM-type scheme for approximating the solution \(U\) of (2.3). The top directed axis represents the coarse grid that holds values of \(U\). In the lower axis, local solutions \(u^\epsilon, v^\epsilon\) to (2.2) are computed using an initial condition determined by \(U(t_n)\). Then, \(\bar{f}\) is evaluated by averaging the solutions with a compactly supported kernel. This procedure is summarized below.

1. Reconstruction: at \(T = t_n\), set the initial conditions for \(u_n^\epsilon, v_n^\epsilon\) using \(U^n\).
2. Microscale evolution: solve (2.2) in a local domain \(t \in [t_n, t_{n+\eta}]\) and use an averaging kernel \(K\) to compute \(\bar{f}(t_n) \sim \tilde{f}(t_n) = K * f(u_n^\epsilon, v_n^\epsilon)\).
3. Macroscale evolution: compute \(U^{n+1}\) at \(T = t_{n+1}\) using \(\{U\}_{j=0}^{n}\) and \(\{\tilde{f}(t_j)\}_{j=0}^{n}\).

Consider the following simple example of the type studied in [8],
\[
\begin{align*}
\frac{dx}{dt} &= y^2, \quad x(0) = 0 \\
\frac{dy}{dt} &= \frac{1}{\epsilon} z, \quad y(0) = 1 \\
\frac{dz}{dt} &= \frac{1}{\epsilon} y, \quad z(0) = 0.
\end{align*}
\]
The solutions are \(x(t) = \frac{t}{2} + \frac{1}{\epsilon} \sin\left(\frac{2t}{\epsilon}\right), y = \cos\left(\frac{t}{\epsilon}\right),\) and \(z(t) = \sin\left(\frac{t}{\epsilon}\right).\) The slow variable \(x(t)\) is of the form given in (1.3).

Nonuniform discretizations are also used in approximation solutions to partial differential equations (PDEs), for example,
\[
\begin{align*}
-\text{div}(a^\epsilon(x) \nabla u^\epsilon(x)) &= f(x) \quad x \in \Omega \subset \mathbb{R}^d \\
u^\epsilon(x) &= 0 \quad x \in \partial \Omega
\end{align*}
\]
Here, $\epsilon$ is a parameter satisfying $0 < \epsilon < 1$ that represents the ratio between the large and small scales in the multiscale conductivity $a^\epsilon(x) := a(x, x/\epsilon)$, where $a(x, y)$ is assumed to be periodic in $y$.

For $a^\epsilon \in L_\infty(\mathbb{R}^d)$ the solution $u^\epsilon$ of (2.5) contains a high frequency component of $O(\epsilon^{-1})$ and a low frequency component of $O(1)$. The problem (2.5) is well studied in homogenization theory [3, 10], and demonstrates the application of these homogenization techniques in multiscale computation.

Numerical methods such as the finite element heterogeneous multiscale method [1, 6] approximate the solution to an effective problem using grids with macroscale spacing $\Delta x > \epsilon$.

The functions involved in homogenization theory and multiscale computation are often of lower regularity. In order to connect to information theory, we will approximate them by bandlimited functions, described in the next section.

2.3. Sampling of bandlimited signals. Bandlimited signals are well studied in sampling theory. The set of functions bandlimited to the bounded set $\mathcal{F} \subset \mathbb{R}^d$ is denoted

$$
\mathcal{B}(\mathcal{F}) = \{ g \in L_2(\mathbb{R}^d) \mid \hat{g}(\xi) = 0 \text{ for all } \xi \not\in \mathcal{F} \}.
$$

The celebrated sampling theorem of Shannon provides a characterization of bandlimited functions in one dimension.

**Theorem 2.1 (Shannon, 1949).** A function $f \in \mathcal{B}([-W,W])$ is completely determined by its uniform samples $f\left(\frac{j}{2W}\right)$, $j \in \mathbb{Z}$.

The corresponding reconstruction formula is

$$
f(x) = \sum_{j=-\infty}^{\infty} f(j/2W) \text{sinc}(2Wx - j),
$$

where $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$. The sampling rate of $1/2W$ is known as the Nyquist rate. Furthermore, the $L_2$ norm of $f$ is given by

$$
\|f\|_{L_2(\mathbb{R})}^2 = \frac{1}{2W} \sum_{j=-\infty}^{\infty} |f(j/2W)|^2.
$$

The sampling theorem has played a crucial role in signal processing and communication and motivated various extensions, including results for nonbandlimited functions, finite sampling sets, irregular sampling sets, and generalized interpolation functions [17, 19].

Let $\mathcal{F} \subset \mathbb{R}$ be a bounded set. For a given sampling strategy, a sampling set $X = \{x_j\}$ is a set of stable sampling for $\mathcal{B}(\mathcal{F})$ if there exists a constant $C > 0$ such that

$$
\int_{-\infty}^{\infty} |g(x)|^2 dx \leq C \sum_j |g(x_j)|^2 \text{ for all } g \in \mathcal{B}(\mathcal{F}).
$$

In [12, 13], Landau obtained a bound on the minimum sampling rate required for the stable reconstruction of a bandlimited function. The Landau rate is the sum of the bandwidths of a function. For multiband functions with large spectral gaps, the Landau rate can be much lower than the Nyquist rate.

The main result in this paper produces a stable set of sampling for a class of multiscale functions with a sampling density that attains the Landau bound.
2.4. Related work. The motivating result for this work is in [7], regarding functions with scaling law representation (1.1) where \( f \in L^2(\mathbb{R}^2) \) is bandlimited and 1-periodic in both variables. It is also assumed that \( \epsilon = 1/L_1 \) for a positive integer \( L_1 \), so that \( f \) can be represented by a finite Fourier series:

\[
f(x, y) = \sum_{n=0}^{N} \sum_{m=0}^{M} f_{n,m} \exp(2\pi i (nx + my)).
\]

Applying the classical Shannon sampling theorem to \( f^\epsilon \) requires a sampling rate of \( O(\epsilon^{-1}) \). In [7], the data points are clustered in groups with \( \Delta x = 1/L_2 \), for \( L_2, L_1/L_2 \) positive integers,

\[
x_{j,k} = j\Delta x + k\delta x \quad 1 \leq j \leq J, \quad 1 \leq k \leq K, \quad (\delta x << \Delta x).
\]

The resulting set of equations

\[
\sum_{n=0}^{N} \sum_{m=0}^{M} f_{n,m} \exp(2\pi i (nx_{j,k} + mx_{j,k}/\epsilon)) = f^\epsilon(x_{j,k})
\]

has a unique solution for \( J > N \) and \( K > M \), if the conditions \( \delta x < \epsilon/M \) and \( \Delta x < 1/N \) are satisfied. This shows that it is possible to take advantage of the special structure of \( f^\epsilon \) and uniquely reconstruct the function from samples taken from nonuniform sampling set with \( O(N) \) density.

As in compressed sensing [4, 16], we are interested in reconstruction methods for signals that are bandlimited and sparse in the frequency domain. The difference here is that we assume \( \text{a priori} \) knowledge of the spectral properties of \( f(x, y) \) in the representation (1.1).

There are very similar nonuniform sampling theorems using bunched samples to reconstruct functions that have spectral gaps [2, 14, 18]. The result presented here differs from these results in terms of conditions on the spectral gaps or the notion of stability.

3. Bandlimited and periodic functions. This section describes properties of functions that are studied in multiscale analysis. We begin with the construction of basic multiscale functions from functions of two variables.

A function \( f \in L^2(\mathbb{R}^2) \) that satisfies (1.2) can be written as a Fourier series for every fixed \( x \in \mathbb{R}^2 \):

\[
\begin{align*}
  f(x, y) &= \sum_{m=-M}^{M} c_m(x) e^{2\pi i m y}, \quad c_m(x) = \int_{-1}^{1} f(x, y) e^{-2\pi i m y} dy.
\end{align*}
\]

It is readily seen that the functions \( c_m, -M \leq m \leq M \) are in the space \( \mathcal{B}((-N, N)) \) by taking the Fourier transform,

\[
\begin{align*}
  \hat{c}_m(\xi) &= \int_{\mathbb{R}} \left( \int_{-1}^{1} f(x, y) e^{-2\pi i m y} dy \right) e^{-2\pi i \xi x} dx \\
  &= \int_{\mathbb{R}} \int_{-1}^{1} f(x, y) e^{-2\pi i (\xi x + my)} dy dx \\
  &= \hat{f}(\xi, m) = 0 \text{ for all } \xi \notin (-N, N).
\end{align*}
\]
Substituting \((x, \xi)\) for \((x, y)\) in (3.1) results in the representation

\[
(3.2) \quad f^\epsilon (x) = \sum_{m=-M}^{M} c_m(x) e^{2\pi i m x / \epsilon}, \quad c_m \in \mathcal{B}((-N, N)).
\]

\[
\hat{f}^\epsilon (\xi) = \sum_{m=-M}^{M} \hat{c}_m \left( \xi - \frac{m}{\epsilon} \right).
\]

Since \(\hat{c}_m(\xi - \frac{m}{\epsilon}) = 0\) for \(\xi \not\in (-N + \frac{m}{\epsilon}, N + \frac{m}{\epsilon})\), it follows that

\[
supp(\hat{f}^\epsilon) \subset \bigcup_{m=-M}^{M} \left(-N + \frac{m}{\epsilon}, N + \frac{m}{\epsilon}\right).
\]

We will extend the result in [7] to functions \(f(x, y) \in L_2(\mathbb{R}^2)\) that are band limited in both variables and 1-periodic in the second variable and formulate a sampling strategy for functions \(f^\epsilon \in L_2(\mathbb{R})\) of the form (3.2). Let \((-W, W)\) be the smallest interval containing the support of \(\hat{f}^\epsilon\). What happens if we sample the function at a smaller rate, \(\frac{1}{\Delta x} < 2W\)? This question will be addressed in the next section using tools from Fourier analysis and the Poisson summation formula.

4. Nonuniform periodic sampling. For the sampling sets \(X_k\) defined by (1.4), we aim to derive sufficient conditions on \(\Delta x\) and \(\delta x\) so that the function \(f^\epsilon\) can be uniquely and stably recovered from the nonuniform samples \(f^\epsilon(z), z \in \cup_k X_k\).

When sampling at sub-Nyquist rate \(\frac{1}{\Delta x} < 2(\frac{M}{\epsilon} + N)\), the unique recovery of \(f^\epsilon\) on the set \(X_k\) is not guaranteed. We can, however, describe the function reconstructed from undersampling.

For \(g \in \mathcal{B}((-N, N))\), the function \(S_{X_k} g\) is defined in [2] by the formula

\[
(4.1) \quad S_{X_k} g(x) = \sum_{y \in X_k} g(y) \varphi_{\Delta x}(x - y); \quad \varphi_{\Delta x}(z) = \Delta x \int_{\frac{\pi}{\Delta x}}^{\frac{\pi}{\Delta x}} e^{2\pi i \xi z} d\xi.
\]

The following result provides the explicit form of the functions reconstructed from sampling the multiscale function \(f^\epsilon\) on the set \(X_k\). We will denote \(L_m \in \mathbb{Z}\), \(\alpha_m \in [0, 1/\Delta x)\) to be the unique constants that satisfy

\[
(4.2) \quad \frac{m}{\epsilon} = \frac{L_m}{\Delta x} + \alpha_m, \quad -M \leq m \leq M.
\]

**Lemma 4.1.** Let \(f^\epsilon\) satisfy (3.2). The function \(S_{X_0} f^\epsilon\) defined by (4.1) is square integrable and has the explicit form

\[
(4.3) \quad S_{X_0} f^\epsilon(x) = \sum_{m=-M}^{M} c_m(x) e^{2\pi i \alpha_m x}.
\]

Moreover, the reconstruction of \(S_{X_0} f^\epsilon\) is stable,

\[
(4.4) \quad \|S_{X_0} f^\epsilon\|_{L^2}^2 \leq \frac{1}{2N} \sum_{y \in X_0} |f^\epsilon(y)|^2.
\]
Proof. First, note that

\[(4.5) \quad S_{X_0}f^\epsilon(x) = \sum_{m=-M}^{M} S_{X_0}\tilde{c}_m(x)\]

where \(\tilde{c}_m(x) = c_m(x)e^{2\pi imx/\epsilon}\). The Shannon sampling theorem (2.6) can be directly applies to \(c_0 \in B(\Omega)\), to obtain the formula \(S_{X_0}c_0(x) = c_0(x) \in L_2(\mathbb{R})\).

For \(m \neq 0\),

\[
S_{X_0}\tilde{c}_m(x) = \sum_{n=-\infty}^{\infty} c_m(n\Delta x)e^{2\pi i\alpha_m n\Delta x} \frac{1}{\Delta x} \int_{\frac{-\pi}{\Delta x}}^{\frac{\pi}{\Delta x}} e^{2\pi i(x-n\Delta x)\xi} d\xi
\]

\[
= \int_{\frac{-\pi}{\Delta x}}^{\frac{\pi}{\Delta x}} \left[ \Delta x \sum_{n=-\infty}^{\infty} c_m(n\Delta x)e^{-2\pi in\Delta x(\xi-\alpha_m)} \right] e^{2\pi ix\xi} d\xi
\]

\[
= \int_{\frac{-\pi}{\Delta x}}^{\frac{\pi}{\Delta x}} \sum_{\xi=-\infty}^{\infty} \tilde{c}_m(\xi - \alpha_m) e^{2\pi i\xi} d\xi
\]

\[
= c_m(x) e^{2\pi i\alpha_m x}
\]

Since \(c_m, \varphi_{\Delta x} \in L_2(\mathbb{R})\), the sums converge uniformly and the exchange between sum and integral is justified. The Poisson summation formula is used for (4.6). Then, substituting (4.7) in (4.5) proves the reconstruction result. Since \(S_{X_0}c_m \in L_2(\mathbb{R})\) for each \(m\), an application of the triangle inequality shows that \(S_{X_0}f^\epsilon\) is square integrable.

For stability,

\[
\|S_{X_0}f^\epsilon\|_{L_2}^2 = \Delta x \sum_{y \in X_0} |f^\epsilon(y)|^2 < \frac{1}{2N} \sum_{y \in X_0} |f^\epsilon(y)|^2.
\]

A similar result holds for uniform samples taken on the shifted set \(X_k, k \neq 0\). Define \(g^\epsilon \in B(\cup_{m=-M}^{M} (-N + \frac{m}{\epsilon}, N + \frac{m}{\epsilon}))\) by the translation \(g^\epsilon(x) = f^\epsilon(x + k\delta x)\). Then, \(S_{X_k}g^\epsilon(x) = S_{X_0}g^\epsilon(x - k\delta x)\). By Lemma 4.1, \(S_{X_0}g^\epsilon(x) = \sum_{m=-M}^{M} g_m(x)e^{2\pi imk\delta x/\epsilon}\) and

\[
(4.8) \quad \|S_{X_k}f^\epsilon\|_{L_2}^2 = \|S_{X_0}g^\epsilon\|_{L_2}^2 < \frac{1}{2N} \sum_{y \in X_0} |g^\epsilon(y)|^2 = \frac{1}{2N} \sum_{y \in X_k} |f^\epsilon(y)|^2.
\]

The change of variables \(x \to x - k\delta x\) results in the expression

\[
(4.9) \quad S_{X_k}f^\epsilon(x) = \sum_{m=-M}^{M} c_m(x)e^{2\pi i(m\alpha x + k\delta x \frac{m}{\alpha})}.
\]

This expression for the functions reconstructed from shifted samples of \(f^\epsilon\) will be used in the proof of the main result.
5. Proof of Theorem 1.1. The stable reconstruction formula for multiscale functions $f^r$ is proved using an approach similar to [7].

Proof. The set of equations (4.9) for $k = -K, \ldots, K$ produces the following linear system

\begin{equation}
\begin{bmatrix}
S_{X_0}f^r(x) \\
S_{X_1}f^r(x) \\
\vdots \\
S_{X_K}f^r(x)
\end{bmatrix}
= 
\begin{bmatrix}
1 & \ldots & 1 & \ldots & 1 \\
w_{-M} & \ldots & w_0 & \ldots & w_M \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
w_{K-M} & \ldots & w_K & \ldots & w_M
\end{bmatrix}
\begin{bmatrix}
C_{-M}(x)e^{2\pi i\alpha_M x} \\
\vdots \\
C_0(x) \\
\vdots \\
C_M(x)e^{2\pi i\alpha_M x}
\end{bmatrix}.
\end{equation}

When $K = M$, the matrix formed in (5.1) is a square Vandermonde matrix $V_M$ with elements $w_m = e^{2\pi i \frac{m}{M} \delta x}$. The requirement that $w_{-M}, \ldots, w_M$ are distinct elements results in an invertible system.

Since $0 < \frac{\delta x}{\epsilon} < \frac{1}{2M+1}$, the elements $w_{-M}, \ldots, w_M$ are distinct nodes distributed on the unit circle, with $w_1, \ldots, w_M$ in the upper half plane and $w_{-M}, \ldots, w_{-1}$ in the lower half plane. This ensures the existence of $V_M^{-1}$. As a result, the reconstruction formula for $f^r$ is well defined:

\begin{equation}
f^r(x) = \sum_{i=-M}^{M} \sum_{j=-K}^{K} (V_M^{-1})_{ij} S_{X_j} f^r(x) e^{2\pi i \frac{j}{M} \frac{x}{\epsilon}}.
\end{equation}

Using some properties of Vandermonde matrices, it can be shown that this formula is stable. We will state a result in [9] concerning the norm of $V_M^{-1}$.

**Theorem 5.1** (Gautschi, 1990). For arbitrary $w_l \in \mathbb{C}$, with $w_l \neq w_{l'}$ if $l \neq l'$, there holds

\begin{equation}
\max_l \prod_{l' \neq l} \max(1, |w_{l'}|) \leq \|V_M^{-1}\|_\infty \leq \max_l \prod_{l' \neq l} \frac{1 + |w_{l'}|}{|w_l - w_{l'}|},
\end{equation}

where $V_M$ is a Vandermonde matrix with elements $w_{-M}, \ldots, w_M \in \mathbb{C}$. The upper bound is obtained if $w_l = |w_l| e^{i\theta_l}, l = -M, \ldots, M$ for some fixed $\theta \in \mathbb{R}$.

The upper bound on $|w_l - w_{l'}|$ can be computed

\begin{equation}
|w_l - w_{l'}| \leq |e^{2\pi i M \delta x / \epsilon} - 1| < |e^{2\pi i \frac{M}{2M+1}} - 1| < 2.
\end{equation}

Now we compute the smallest distance between adjacent nodes. In the first case, for $-M \leq l \leq M$,

\[ |w_{l+1} - w_l| = |e^{2\pi i \frac{1}{\epsilon} - (\alpha_{l+1} - \alpha_l)} \delta x - 1| > |e^{2\pi i \frac{1}{\epsilon} - \Delta x} \delta x - 1| \]

We need to also consider the distance between adjacent nodes $w_{-M}$ and $w_M$. Using the fact that $\min(2M \delta x / \epsilon, 1 - 2M \delta x / \epsilon) > \delta x / \epsilon$,

\[ |w_{-M} - w_M| \geq |e^{2\pi i 2M \delta x / \epsilon} - 1| > |e^{2\pi i \delta x / \epsilon} - 1| > |e^{2\pi i (\frac{1}{2} - \Delta x) \delta x - 1}|. \]

Since $\Delta x < \frac{1}{\epsilon}$, the last term in both cases is nonzero. Applying this to (5.1) results in the estimate

\[ \frac{1}{2M+1} < \|V_M^{-1}\|_\infty \leq \frac{1}{|e^{2\pi i (\frac{1}{2} - \Delta x) \delta x - 1}| 2M+1}. \]
Now we have a final stability estimate for the reconstruction of \( f^\epsilon \) from sampling sets \( X_k, -M \leq k \leq M \),

\[
\| f^\epsilon \|_2^2 \leq \| V_M^{-1} \|_\infty \sum_{k=-M}^{M} S_{X_k} f^\epsilon(x) \|_2^2 < C(\delta x/\epsilon) \sum_{y \in \cup X_k} | f^\epsilon(y) |^2,
\]

where \( C(\delta x/\epsilon) = \frac{2M+1}{2N} \left( \exp(2\pi i(\frac{1}{2} - \Delta x)\delta x) - 1 \right)^{-(2M+1)} \).

6. Localized microscale. Multiscale models can be divided into different groups based on common features of the problems. In [1, 6], type A problems are described as problems that require microscale resolution with a microscale solver in a fixed number of local domains. This could be in order to resolve contain isolated defects such as dislocations, cracks, shocks, and contact lines. Outside of these local domains, the macroscale solver is used. Type B problems require some microscale information throughout the entire computational domain. Localized microscale simulations are used to supply missing information to the microscale solver.

The sampling analysis in the earlier sections have referred to type B problems. There are also links to information theory for type A problems involving functions of the following form

\[
\lim_{|t| \to \infty} f(t) = \begin{cases} 
  f^+(t), & t > 0 \\
  f^-(t), & t < 0,
\end{cases}
\]

where \( f^+, f^- \) are branches of bandlimited function. We will assume the specific form \( f(\gamma^{-1}(\epsilon)) \), where the transformation \( \gamma \) models the behavior of the isolated defect. Adaptive mesh refinement techniques are designed for these problems to provide higher resolution near a singularity or a domain of strong variation. These types of discretizations match well with the sampling results in [5, 11] for time warped signals.

In [5], the exact reconstruction of \( f \) from irregularly spaced samples \( \{t_n\} \) is guaranteed, provided that there exists a continuous, injective mapping \( t_n \to \gamma(t_n) \). The main result for time warped one dimensional signals is the following theorem.

**Theorem 6.1.** Let a function \( f(t) \) of one variable be sampled at the points \( t = t_n, n \in \mathbb{Z} \), where \( t_n \) is not necessarily a sequence of uniformly spaced numbers. If a 1-1 continuous mapping \( \gamma(t) \) exists such that \( n/2W = \gamma(t_n) \), and if \( h(\tau) = f(\gamma^{-1}(\tau)) \) is bandlimited to \([-W, W] \), then the following equation holds

\[
\sum_{n=-\infty}^{\infty} f(t_n) \frac{\sin(2W\gamma(t) - n)}{2W\gamma(t) - n}.
\]

As an example, we set \( f^\epsilon(t) = \sin(t + 2 \arctan(t/\epsilon)) \). Then, \( f^+ (t) = \sin(t + \pi) \) and \( f^- (t) = \sin(t - \pi) \). Define \( \gamma^{-1}_\epsilon : \mathbb{R} \to \mathbb{R} \) by the transformation

\[
\gamma^{-1}_\epsilon (t) = t + \arctan(t/\epsilon).
\]

Since \( \gamma^{-1}_\epsilon \) is surjective and \( (\gamma^{-1}_\epsilon)'(t) = 1 + \frac{e^{-1}}{\sqrt{1 + (te^{-1})^2}} > 0 \), it follows that \( \gamma_\epsilon(t) := (\gamma^{-1}_\epsilon)^{-1}(t) \) is well defined.
Then, there exists a set of sampling points \( t_n, n \in \mathbb{Z} \) such that \( \gamma_n(t_n) = n \). The function \( h(\tau) = f(\gamma(\tau)) \) is a bandlimited function that can be reconstructed from the uniform samples at points \( \tau_n = \gamma_n(t_n) \).

Therefore, the original function can be recovered using the relation \( f(t) = h(\gamma^{-1}(t)) \), and we have shown that recovery of \( f \) is possible using the nonuniform sampling set \( \gamma_n(t_n), n \in \mathbb{Z} \).

There are cases where the explicit form of the transformation \( \gamma \) is unknown. In [11], a local estimate \( B(t) \) of the effective bandwidth of \( f \) is made using techniques such as the windowed Fourier transform. A disadvantage here is that the “local” bandwidth cannot be extended globally.

We have that a signal with time varying bandwidth \( B(t) \) can be defined as

\[
    f(t) = \sin(2\pi \phi(t))
\]

where the phase function \( \phi(t) \) is defined in terms of the instantaneous signal frequency by \( \phi'(t) = B(t) \). The samples required for an exact reconstruction are given implicitly

\[
    t_n = n/(2B(t_n)).
\]

In the above example, this corresponds to a sampling set of the form

\[
    t_n = \frac{n\pi}{t_n + 2 \arctan(t_n/\epsilon)}.
\]

7. Conclusions. Multiscale computations aim to design numerics for the simulations of coupled models on different scales. In the theory of homogenization of differential equations, multiscale processes are described by the combination of models on different scales. The relevant functions in the homogenization of differential equations and in the theory of heterogeneous multiscale methods satisfy the scaling law, \( f^\epsilon(x) = f(x, x/\epsilon) \), where \( f(x, y) \) is periodic in the second variable and the parameter \( \epsilon << 1 \) represents the ratio of scales in the problem.

We view these functions in the setting of information theory by making the further assumption that \( f \) belongs to the class of bandlimited functions and has a Fourier decomposition

\[
    f(x, y) = \sum_k f_k(x) e^{2\pi iky}.
\]

A solver coupling the macro and micro scales would find a compromise between the effective solution on a macroscale grid of size \( \Delta x \) and the full direct numerical simulation on a fine scale grid with spacing \( \delta x << \Delta x \). We show that a computational grid designed to fully resolve \( f^\epsilon \) should be of the form

\[
    f^\epsilon(z), \quad z \in \{ j\Delta x + k\delta x \mid j \in \mathbb{Z}, -K \leq k \leq K \},
\]

which matches sampling strategies used in information theory when applied to bandlimited signals.

The so called type A problem in HMM is also discussed. The microscale is here only relevant in one or a few locations. For these problems, there is also a close connection between mesh refinement and information theory.

Multiscale modeling and sampling strategies in higher dimensions will be discussed in a forthcoming paper.
REFERENCES


