ROTATIONAL INVARIANCE OF QUADROMER CORRELATIONS ON THE HEXAGONAL LATTICE

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Abstract. In 1963 Fisher and Stephenson [FS] conjectured that the monomer-monomer correlation on the square lattice is rotationally invariant. In this paper we prove a closely related statement on the hexagonal lattice. Namely, we consider correlations of two quadromers (four-vertex subgraphs consisting of a monomer and its three neighbors) and show that they are rotationally invariant.

1. Introduction

The lines of the square grid divide the plane into unit squares called *monomers*. A *dimer* is the union of two monomers that share an edge, and a collection of disjoint dimers is said to be a *dimer tiling* of the planar region obtained as their union. We denote by $\mathrm{M}(R)$ the number of dimer tilings of the region R.

Let S_n be the square $[0,2n]\times[0,2n]$. Denote by $S_n(r,s;p,q)$ the region obtained from S_n by removing the two monomers whose lower left corners have coordinates (r,s) and $(r+p,s+q)$, $0\leq r,s,r+p,s+q\leq 2n-1$.

The *boundary-influenced* correlation of the two removed monomers is defined in [**FS**] as

$$\omega_b(r,s;p,q):=\lim_{n\to\infty}\frac{\mathrm{M}(S_n(r,s;p,q))}{\mathrm{M}(S_n)}. \tag{1.1}$$

If the monomers of S_n are colored in a chessboard fashion, every dimer consists of one monomer of each color, and one sees that $\omega_b(r,s;p,q)=0$ unless the two removed monomers have opposite colors. Therefore in the following discussion we assume that the latter condition holds (this amounts to p and q having opposite parities).

The correlation at the *center* is then defined as

$$\omega(p,q):=\lim_{r,s\to\infty}\omega_b(r,s;p,q). \tag{1.2}$$

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The results in [**FS**] provide explicit exact values for $\omega(p,q)$ for small, concrete values of p and q. In the cases $q=0$ (or $q=1$) and $p=q+1$—which correspond to a lattice and a lattice diagonal direction, respectively—, Fisher and Stephenson [**FS**] computed the values of these correlations for the first thirteen (respectively, first 22) values of p. These tables provide strong evidence that the correlations $\omega(p,q)$ decay to zero at exactly the same rate (namely, as the inverse square root of the distance between the monomers, with exactly the same constant of proportionality) along this two inequivalent directions[1]. After making this remark, Fisher and Stephenson continue in [**FS**] by stating: "This equality suggests that the monomer correlations decay isotropically (with full angular symmetry)." This conjecture was the starting point of the present paper.

Clearly, these considerations can also be made on the triangular lattice. A monomer is then a unit equilateral triangle, and the dimers are the unit rhombi consisting of the union of two monomers with a common edge.

Each dimer covers one up-pointing and one down-pointing monomer, so the two orientations of monomers play in this case the role of the two colors in the chessboard coloring of the square lattice. Therefore, the natural analog of the conjecture of Fisher and Stephenson would be the statement that the correlation of two monomers of opposite orientation is rotationally invariant.

A *quadromer* is by definition the union of any monomer with the three monomers adjacent to it. Clearly, this is just a lattice triangle of side two.

It is correlations of two quadromers that we prove are rotationally invariant in this paper. Our method of proof is based on exact counting of dimer coverings of certain regions on the triangular lattice, and the reduction of the problem to these exact counts works only when we remove triangles of side two—quadromers—, and not when monomers are removed. In essence, this is due to the fact that two is an even number, and this allows to express the relevant quantities as convenient determinants, evaluate them and obtain the asymptotics of the resulting expressions.

We note that removing a quadromer from a region is equivalent to removing any two of its outer monomers—such a pair could be called a *bimer*. Indeed, any dimer covering of the region resulting by removal of a bimer must contain the unit rhombus fitting in its notch, thus extending it to a quadromer.

In order to define quadromer correlations, we need a family of regions on the triangular lattice to play the role the squares S_n played on the square lattice. The choice of these regions is perhaps the single most crucial part in our proof.

To this end, for any integer $n \geq 1$ we define the region P_n as follows. Fix a vertex O of the triangular lattice and consider the zig-zag lattice path L_u of length $2n$ extending upward from O and taking alternate steps northeast and northwest (see Figure 1.1). Let O' be the southwestern neighbor of O, and consider a second zig-zag lattice path L_d of length $2n$, extending downward from O' and taking alternate steps southeast and southwest. The union of these two lattice paths and the segment OO' forms the eastern boundary of P_n. The rest of its boundary is defined by moving from the lowest vertex of L_d successively

[1] Hartwig [**H**] proved that the correlations decay as predicted by Fisher and Stephenson along the lattice diagonal direction.

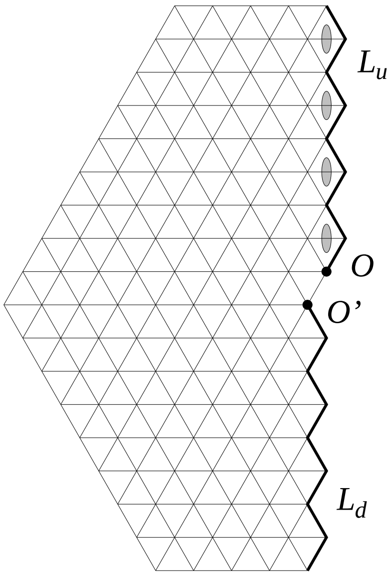

FIGURE 1.1. P_4.

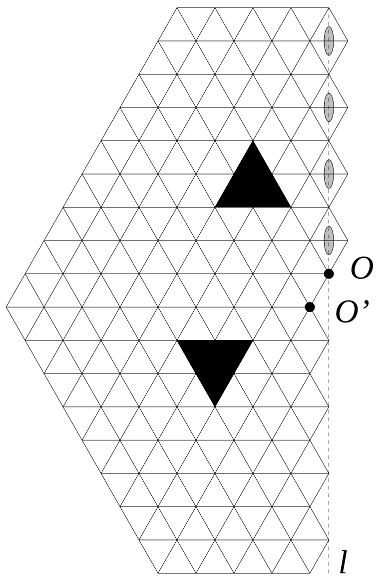

FIGURE 1.2. $P_4(3,0;2,1)$.

n units west, $2n$ units northwest, $2n+1$ units northeast and n units east, along lattice lines throughout, to arrive at the topmost vertex of L_u. Our region P_n is the region traced out this way, with the additional requirement that the n dimer positions closest to L_u have weight $1/2$ (these positions are indicated by shaded ellipses in Figure 1.1): what this means is that a dimer covering of P_n using precisely k of these dimers gets weight $1/2^k$, and $\mathrm{M}(P_n)$ is the *weighted* count of the dimer coverings of P_n, i.e., the sum of all their weights.

Let ℓ be the vertical line through O. Denote by $D(R,v)$ be the down-pointing quadromer whose base is centered R units to the left of ℓ and lies on a horizontal lattice line crossing L_d $2v+1$ units below O'. Let $U(R,v)$ be the up-pointing quadromer whose base is centered R units to the left of ℓ and lies on a horizontal lattice line crossing L_u $2v$ units above O (Figure 1.2 shows quadromers $D(3,0)$ and $U(2,1)$)[2] .

Let $P_n(R_1,v_1;R_2,v_2)$ be the region obtained from P_n by removing the quadromers $D(R_1,v_1)$ and $U(R_2,v_2)$, where $R_1,R_2\geq 1$ and $v_1,v_2\geq 0$ (see Figure 1.2 for an example).

Paralleling (1.1), we define the boundary-influenced correlation of two removed quadromers as

$$\omega_b(R_1,v_1;R_2,v_2) := \lim_{n\to\infty} \frac{\mathrm{M}(P_n(R_1,v_1;R_2,v_2))}{\mathrm{M}(P_n)}. \tag{1.3}$$

In analogy to (1.2), we define the correlation of the quadromers at the center, with

[2]Considering the relative position to L_u and L_d, there are two distinct types of both up-pointing and down-pointing quadromers. The reason we made the indicated choice out of the total of four possibilities is that it provides even vertical separations between the bases of the quadromers. The other three choices would entail only minor alterations to the considerations in this paper.

horizontal separation $r \geq 0$ and vertical separation $\sqrt{3}u \geq \sqrt{3}$, by

$$\omega(r, u) := \lim_{R \to \infty} \omega_b(R + r, u - 1; R, 0). \tag{1.4}$$

When making the separation (r, u) of the two quadromers grow to infinity it is natural to do it so that $u = qr + c$, where q and c are fixed rational numbers.

The main theorem of this paper is the following.

THEOREM 1.1. *As r and u approach infinity so that $u = qr + c$, with $q \geq 0$ and c fixed rational numbers ($c \geq 0$ if $q = 0$),*

$$\omega(r, u) = \frac{3}{4\pi^2(r^2 + 3u^2)} + o(r^{-2}). \tag{1.5}$$

Since the parenthesis in the denominator in (1.5) is just the square of the distance between the centers of the bases of the removed quadromers, this theorem shows in particular that the quadromer correlation $\omega(r, u)$ is rotationally invariant.

2. A quadruple sum for quadromer correlations

The reason we chose the regions P_n as above is that they have, as shown in [**C**], the remarkable property that quite general alterations of their eastern boundary create regions whose weighted dimer counts are given by simple product formulas.

To state this precisely, view the lattice paths L_u and L_d on the eastern boundary as consisting of n *bumps* each—successions of two lattice steps forming an agle opening to the left. Label the bumps on each path succesively by $0, 1, \dots, n - 1$, starting from the bumps closest to O[3] (see Figure 2.1). Any bump of L_u is allowed to be "removed" by placing an up-pointing quadromer across it and discarding the three monomers of P_n it covers. Similarly, one can remove a bump from L_d by placing a down-pointing quadromer across it and discarding the three monomers of P_n covered by it (these bump removals are illustrated in Figure 2.1).

Let $P_n[k_1, k_2; l_1, l_2]$ be the region obtained from P_n by removing bumps k_1 and k_2 from L_d and bumps l_1 and l_2 from L_u, where $0 \leq k_1 < k_2 \leq n - 1$ and $0 \leq l_1 < l_2 \leq n - 1$ (Figure 2.1 shows $P_4[0, 2; 1, 2]$). In [$\mathbf{C}$, (2.2), (1.1)–(1.6)] explicit simple product formulas (i.e., with factors of size at most linear in the parameters) are given for the weighted count of dimer coverings of a family of regions $\bar{R}_{\mathbf{l},\mathbf{q}}(x)$ that includes the $P_n[k_1, k_2; l_1, l_2]$'s ($\mathbf{l}$ and $\mathbf{q}$ are lists of strictly increasing positive integers, and x is a nonnegative integer; in the notation of [**C**], $\bar{R}_{[1,\dots,k_1,k_1+2,\dots,k_2,k_2+2,\dots,n],[1,\dots,l_1,l_1+2,\dots,l_2,l_2+2,\dots,n]}(n)$ is the region that we denote here $P_n[k_1, k_2; l_1, l_2]$). Thus we obtain simple product formulas for the numbers $\mathrm{M}(P_n[k_1, k_2; l_1, l_2])$. Using them, one obtains after straightforward if somewhat lengthy

[3]Note that this differs from the labeling used in [**C**], which is obtained by increasing the present labels by 1.

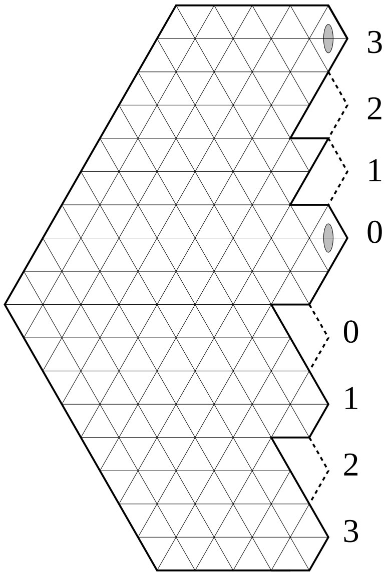

FIGURE 2.1. $P_4[0,2;1,2]$.

manipulations that

$$\lim_{n\to\infty} \frac{\mathrm{M}(P_n[k_1,k_2;l_1,l_2])}{\mathrm{M}(P_n)} = \frac{2^{-4}(2k_1+1)!\,(2k_2+1)!\,(2l_1+1)!\,(2l_2+1)!}{2^{2k_1+2k_2+2l_1+2l_2}k_1!\,(k_1+1)!\,k_2!\,(k_2+1)!\,l_1!^2\,l_2!^2} \times \frac{(k_2-k_1)(l_2-l_1)}{(k_1+l_1+2)(k_1+l_2+2)(k_2+l_1+2)(k_2+l_2+2)}. \tag{2.1}$$

LEMMA 2.1. *The boundary-influenced correlation $\omega_b(R_1,v_1;R_2,v_2)$ is given by*

$$\begin{aligned}\omega_b(R_1,v_1;R_2,v_2) =& 2^{-4}R_1R_2(R_2-1/2)(R_2+1/2)\\ &\times\Big|\sum_{a,b=0}^{R_1}\sum_{c,d=0}^{R_2}(-1)^{a+b+c+d}\frac{(R_1+a-1)!\,(R_1+b-1)!}{(2a)!\,(R_1-a)!\,(2b)!\,(R_1-b)!}\\ &\times\frac{(R_2+c-1)!\,(R_2+d-1)!}{(2c+1)!\,(R_2-c)!\,(2d+1)!\,(R_2-d)!}\\ &\times\frac{(2v_1+2a+1)!\,(2v_1+2b+1)!}{2^{2(2v_1+a+b)}(v_1+a)!\,(v_1+a+1)!\,(v_1+b)!\,(v_1+b+1)!}\\ &\times\frac{(2v_2+2c+1)!\,(2v_2+2d+1)!}{2^{2(2v_2+c+d)}(v_2+c)!^2\,(v_2+d)!^2}\\ &\times\frac{(b-a)^2(d-c)^2}{(u+a+c)(u+a+d)(u+b+c)(u+b+d)}\Big|,\end{aligned} \tag{2.2}$$

where $u = v_1+v_2+2$.

To prove this Lemma we will need the following special case of the Lindström-Gessel-Viennot theorem on non-intersecting lattice paths.

Our lattice paths will be paths on the directed grid graph $\mathbb{Z}^2$, with edges oriented so that they point in the positive direction. We allow the edges of $\mathbb{Z}^2$ to be weighted, and define the weight of a lattice path to be the product of the weights on its steps. The weight of an N-tuple of lattice paths is the product of the individual weights of its members. The weighted count of a set of N-tuples of lattice paths is the sum of the weights of its elements.

Let $\mathbf{u} = (u_1, \dots, u_N)$ and $\mathbf{v} = (v_1, \dots, v_N)$ be two fixed sets of starting and ending points on $\mathbb{Z}^2$, and let $\mathcal{N}(\mathbf{u}, \mathbf{v})$ be the set of non-intersecting lattice paths with these starting and ending points. For $\mathbf{P} \in \mathcal{N}(\mathbf{u}, \mathbf{v})$, let $\sigma_{\mathbf{P}}$ be the permutation induced by $\mathbf{P}$ on the set consisting of the N indices of its starting and ending points.

Theorem 2.2 (Lindström-Gessel-Viennot).

$$\sum_{\mathbf{P} \in \mathcal{N}(\mathbf{u}, \mathbf{v})} (-1)^{\sigma_{\mathbf{P}}} \operatorname{wt}(\mathbf{P}) = \det\left((a_{ij})_{1 \leq i,j \leq n}\right),$$

where a_{ij} is the weighted count of the lattice paths from u_i to v_j.

What makes possible the use of this result in our setting is a well-known procedure of encoding dimer coverings by families of non-intersecting "paths of dimers:" given a dimer covering T of a region R on the triangular lattice and a lattice line direction d, the dimers of T parallel to d (i.e., having two sides parallel to d) can naturally be grouped into non-intersecting paths joining the lattice segments on the boundary of R that are parallel to d, and conversely this family of paths determines the dimer covering (see Figure 2.2 for an illustration of this and e.g. [**C**] for a more detailed account).

We will find it convenient to view the paths of dimers directly as lattice paths on $\mathbb{Z}^2$, thus bypassing the "extra steps" of bijecting them with lattice paths on a lattice of rhombi with angles of 60 and 120 degrees, and then deforming this to the square lattice. In this context, the "points" of our lattice $\mathcal{L}$ are the edges of the triangular lattice parallel to a chosen lattice line direction d—we call them *segments*—, and the "lines" of $\mathcal{L}$ are sequences of adjacent dimers extending along the two lattice line directions different from d. The dimers that these lines consist of are the *edges* of $\mathcal{L}$.

Proof of Lemma 2.1. Choose the lattice line direction d in the above encoding procedure to be the southwest-northeast direction, and choose the positive directions in the lattice $\mathcal{L}$ so that they point east and southeast. Encode the dimer coverings of $P_n(R_1, v_1; R_2, v_2)$ by $(2n+3)$-tuples of non-intersecting paths consisting of dimers parallel to d (see Figure 2.2; there and in the following figures the dimer positions weighted by $1/2$ are not distinguished, but are understood to carry that weight).

Let $\mathbf{P}$ be such a $(2n+3)$-tuple. Consider the permutation $\sigma_{\mathbf{P}}$ induced by $\mathbf{P}$ on the set of the $2n+3$ indices of its starting and ending points. We claim that the sign of $\sigma_{\mathbf{P}}$ is independent of $\mathbf{P}$. Indeed, denote by U and D the removed up-pointing and down-pointing quadromers, respectively. While the way in which the starting and ending points of $\mathbf{P}$—clearly independent of $\mathbf{P}$—are matched up depends on $\mathbf{P}$, it is always the case that the two paths ending on U start at consecutive starting points, and the two starting at D

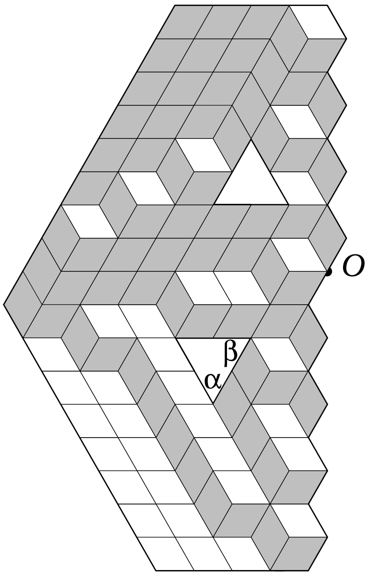

FIGURE 2.2.
A dimer covering of $P_4(3, 0; 2, 1)$
encoded by paths of rhombi.

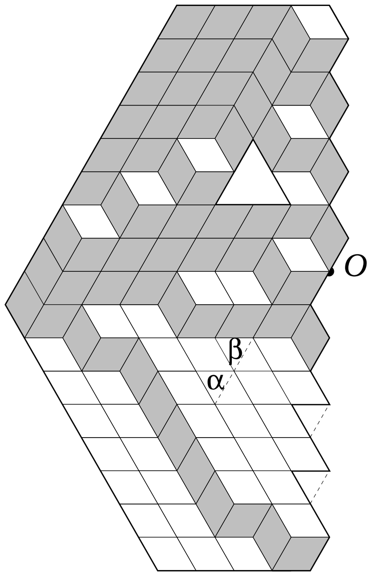

FIGURE 2.3.
The effect of Laplace expansion over
the rows indexed by α and β.

end at consecutive ending points. It is easy to see that this implies that all $\sigma_{\mathbf{P}}$'s have the same sign.

Weight by 1/2 the edges of our "path-encoding" latice $\mathcal{L}$ corresponding to dimer positions weighted by 1/2 in $P_n(R_1, v_1; R_2, v_2)$. Weight all other edges of $\mathcal{L}$ by 1. Then the weight of the dimer covering encoded by $\mathbf{P}$ is just $\mathrm{wt}(\mathbf{P})$, and we obtain by Theorem 2.2 and the constancy of the sign of $\sigma_{\mathbf{P}}$ that

$$\mathrm{M}(P_n(R_1, v_1; R_2, v_2)) = |\det A|\,, \tag{2.3}$$

where A is the $(2n+3) \times (2n+3)$ matrix recording the weighted counts of the lattice paths with given starting and ending points (note that the right hand side of (2.3) is independent of the ordering of these starting and ending points).

We deduce (2.2) by applying Laplace expansion to the determinant in (2.3). Recall that for any $m \times m$ matrix M and any s-subset S of $[m] := \{1, \dots, m\}$, Laplace expansion along the rows with indices in S states that

$$\det M = \sum_K (-1)^{\epsilon(K)} \det M_S^K \det M_{[m]\setminus S}^{[m]\setminus K}\,, \tag{2.4}$$

where K ranges over all s-subsets of $[m]$, $\epsilon(K) := \sum_{k \in K}(k-1)$ and M_I^J is the submatrix of M with row-index set I and column-index set J.

The rows and columns of the matrix A in (2.3) are indexed by the starting and ending points of the $(2n+3)$-tuples of non-intersecting lattice paths encoding the dimer coverings of $P_n(R_1, v_1; R_2, v_2)$. The starting points are the $2n+1$ unit segments along the northwestern boundary of $P_n(R_1, v_1; R_2, v_2)$, together with the two segments α and β of D parallel to d (see Figure 2.2). The ending points are the $2n+1$ segments parallel to d on the eastern boundary of $P_n(R_1, v_1; R_2, v_2)$, together with two more such segments on U.

Apply Laplace expansion to the matrix A of (2.3) along the two rows indexed by α and β (see Figure 2.2). The first determinant in the summand in (2.4) is then just a two by two determinant. Its entries are weighted counts of lattice paths on $\mathcal{L}$ that start at α or β and end at some segment on L_d. There are only R_1+1 segments on L_d that can be reached this way. Label them consecutively from top to bottom by $0, 1, \dots, R_1$. We can restrict summation in (2.4) to the two-element subsets K of this set of segments: all other terms have at least one zero column in the two by two determinant. Therefore we obtain from (2.3) that

$$\mathrm{M}(P_n(R_1, v_1; R_2, v_2)) = \left| \sum_{0\leq a,b\leq R_1} (-1)^{a+b} \det A^{\{a,b\}}_{\{\alpha,\beta\}} \det A^{[2n+3]\setminus\{a,b\}}_{[2n+3]\setminus\{\alpha,\beta\}} \right|. \tag{2.5}$$

Choosing the origin of $\mathcal{L}$ to be at α, one sees that β has coordinates $(-1, 1)$ and the segment labeled j on L_d has coordinates $(R_1 - 1 - j, 2j)$, $j = 0, \dots, R_1$. Since the lattice paths counted by the entries of $A^{\{a,b\}}_{\{\alpha,\beta\}}$ have all steps weighted by 1, the determinant of this matrix is

$$\det A^{\{a,b\}}_{\{\alpha,\beta\}} = \det \begin{bmatrix} \binom{R_1-1+a}{2a} & \binom{R_1-1+b}{2b} \\ \binom{R_1-1+a}{2a-1} & \binom{R_1-1+b}{2b-1} \end{bmatrix} = 2R_1 \frac{(b-a)(R_1+a-1)!\,(R_1+b-1)!}{(2a)!\,(R_1-a)!\,(2b)!\,(R_1-b)!}. \tag{2.6}$$

On the other hand, the second determinant in the summand in (2.5) can be interpreted as being the weighted count of dimer coverings of the region $P^{[a,b]}_n(R_2, v_2)$ obtained from $P_n(R_1, v_1; R_2, v_2)$ by placing back quadromer D and removing the two monomers that contain the segments corresponding to a and b on L_d (see Figure 2.3 for an illustration). Indeed, the Lindström-Gessel-Viennot matrix of this region is precisely $A^{[2n+3]\setminus\{a,b\}}_{[2n+3]\setminus\{\alpha,\beta\}}$, and by the argument that proved (2.3) we obtain that $\mathrm{M}(P^{[a,b]}_n(R_2, v_2))$ is equal to $\det A^{[2n+3]\setminus\{a,b\}}_{[2n+3]\setminus\{\alpha,\beta\}}$, up to a sign that is independent of a and b (indeed, the permutations $\sigma_{\mathbf{P}}$ that occur when applying Theorem 2.2 to the region $P^{[a,b]}_n(R_2, v_2)$ are indepentent of a and b). Therefore, using (2.6) we can rewrite (2.5) as

$$\mathrm{M}(P_n(R_1, v_1; R_2, v_2)) = \\ 2R_1 \left| \sum_{0\leq a<b\leq R_1} (-1)^{a+b} \frac{(b-a)(R_1+a-1)!\,(R_1+b-1)!}{(2a)!\,(R_1-a)!\,(2b)!\,(R_1-b)!} \mathrm{M}(P^{[a,b]}_n(R_2, v_2)) \right|. \tag{2.7}$$

In turn, $\mathrm{M}(P^{[a,b]}_n(R_2, v_2))$ can be expressed by a formula similar to the one above. To obtain this, encode the tilings of $P^{[a,b]}_n(R_2, v_2)$ by lattice paths, choosing this time the

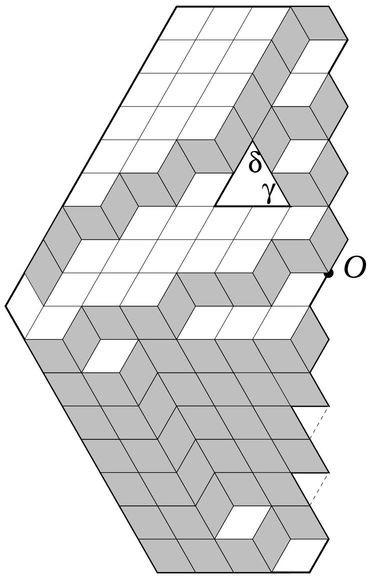

FIGURE 2.4.
A dimer covering of $P_4^{[1,2]}(2,1)$
encoded by paths of rhombi.

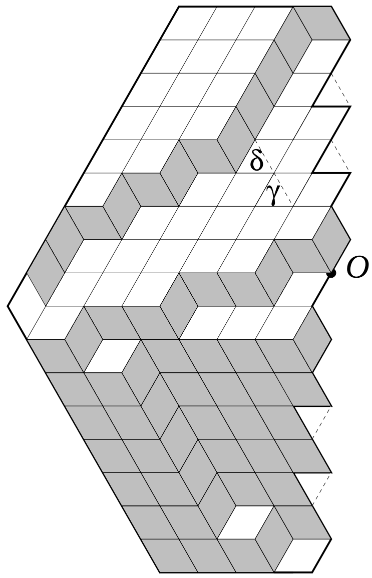

FIGURE 2.5.
The effect of Laplace expansion over
the rows indexed by γ and δ.

lattice direction d to be the southeast-northwest direction, and the positive directions in the encoding lattice $\mathcal{L}$ to point east and northeast. As in the previous encoding, weight by $1/2$ those segments of $\mathcal{L}$ that correspond to dimer positions weighted $1/2$ in $P_n^{[a,b]}(R_2, v_2)$, and weight all its remaining segments by 1.

Each tiling of $P_n^{[a,b]}(R_2, v_2)$ gets encoded this way by a $(2n+2)$-tuple $\mathbf{P}$ of non-intersecting lattice paths on $\mathcal{L}$, starting at the unit segments on its southwestern boundary or at the unit segments γ and δ of U that are parallel to d, and ending at the unit segments parallel to d on its eastern boundary (see Figure 2.4). As in the argument that proved (2.3), the sign of the permutation $\sigma_{\mathbf{P}}$ is independent of $\mathbf{P}$. Therefore, we obtain by Theorem 2.2 that

$$\mathrm{M}(P_n^{[a,b]}(R_2, v_2)) = \epsilon \det B, \tag{2.8}$$

where B is the $(2n+2) \times (2n+2)$ matrix recording the weighted counts of the lattice paths with specified starting and ending points, and the sign ϵ in front of the determinant is the same for all choices of a and b.

Apply Laplace expansion in $\det B$ along the two rows indexed by γ and δ. The first determinant in the summand of (2.4) is again two by two, and records weighted counts of lattice paths starting at γ or δ and ending at some segment on L_u (see Figure 2.4). There are $R_2 + 1$ segments on L_u that can be reached this way; label them consecutively from bottom to top by $0, 1, \dots, R_2$. As with our previous Laplace expansion, we can restrict the summation range in (2.4) to obtain

$$\mathrm{M}(P_n^{[a,b]}(R_2, v_2)) = \epsilon \sum_{0 \le c < d \le R_1} (-1)^{c+d-1} \det B_{\{\gamma,\delta\}}^{\{c,d\}} \det B_{[2n+2]\setminus\{\gamma,\delta\}}^{[2n+2]\setminus\{c,d\}}. \tag{2.9}$$

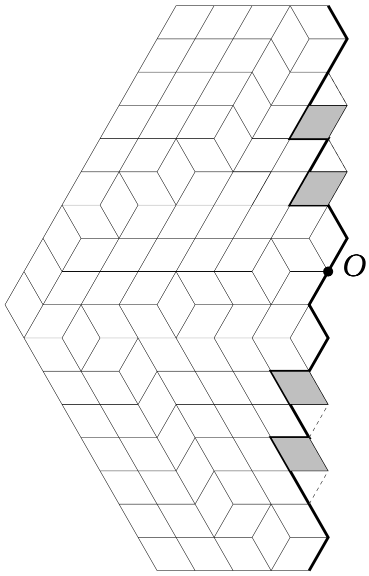

FIGURE 2.6.
The regions $P_4^{[1,2][0,1]}$ and $P_4[1,2;1,2]$
differ only by four dimers.

Centering $\mathcal{L}$ at γ, δ has coordinates $(-1,1)$ and the segment labeled j on L_u has coordinates $(R_2-1-j, 2j+1)$, $j=0,1,\dots,R_2$. The weighted counts involved in the entries of $B_{\{\gamma,\delta\}}^{\{c,d\}}$ (which involve this time some steps of weight $1/2$) are easily calculated and one obtains

$$\det B_{\{\gamma,\delta\}}^{\{c,d\}} = \det \begin{bmatrix} \frac{1}{2}\binom{R_2-1+c}{2c} + \binom{R_2-1+c}{2c+1} & \frac{1}{2}\binom{R_2-1+d}{2d} + \binom{R_2-1+d}{2d+1} \\ \frac{1}{2}\binom{R_2-1+c}{2c-1} + \binom{R_2-1+c}{2c} & \frac{1}{2}\binom{R_2-1+d}{2d-1} + \binom{R_2-1+d}{2d} \end{bmatrix}$$
$$= 2R_2(R_2-1/2)(R_2+1/2)\frac{(d-c)(R_2+c-1)!\,(R_2+d-1)!}{(2c+1)!\,(R_2-c)!\,(2d+1)!\,(R_2-d)!}. \tag{2.10}$$

On the other hand, by applying Theorem 2.2 one more time one sees that

$$\det B_{[2n+2]\setminus\{\gamma,\delta\}}^{[2n+2]\setminus\{c,d\}} = \epsilon' \,\mathrm{M}(P_n^{[a,b][c,d]}), \tag{2.11}$$

where $P_n^{[a,b][c,d]}$ is the region obtained from $P_n^{[a,b]}(R_2, v_2)$ by placing back quadromer U and removing the two monomers near L_u containing segments c and d, and the sign ϵ' is independent of c and d. Furthermore, $P_n^{[a,b][c,d]}$ differs from the region $P_n[v_1+a, v_1+b; v_2+c, v_2+d]$ considered at the beginning of this section only in that the former contains four more dimers, which are weighted by 1 and forced to be part of all its dimer coverings (see Figure 2.6); so the two regions have equal weighted counts of dimer coverings. Therefore, by (2.7), (2.9), (2.10) and (2.11) we obtain that

$$
\begin{aligned}
\mathrm{M}(P_n(R_1,v_1;R_2,v_2)) &= 4R_1R_2(R_2-1/2)(R_2+1/2)\\
\times \Bigg| \sum_{0\le a<b\le R_1}\ \sum_{0\le c<d\le R_1} &(-1)^{a+b+c+d}\frac{(b-a)(R_1+a-1)!\,(R_1+b-1)!}{(2a)!\,(R_1-a)!\,(2b)!\,(R_1-b)!}\\
&\times \frac{(d-c)(R_2+c-1)!\,(R_2+d-1)!}{(2c+1)!\,(R_2-c)!\,(2d+1)!\,(R_2-d)!}\\
&\times \mathrm{M}(P_n[v_1+a,v_1+b;v_2+c,v_2+d])\Bigg|.
\end{aligned}
\tag{2.12}
$$

Dividing (2.12) by $\mathrm{M}(P_n)$, letting $n\to\infty$ and using (2.1), one obtains an expression for $\omega_b(R_1,v_1;R_2,v_2)$ as a quadruple sum in which the summation indices need to satisfy $a<b$ and $c<d$.

The fortunate situation is that, on the one hand, when $a=b$ or $c=d$ the summand becomes zero, and on the other, (2.12) and (2.1) combine to produce a summand which is invariant under independently transposing a with b and c with d (because the differences $b-a$ and $d-c$ end up appearing at the second power). Therefore the summation range may be extended to the one shown in (2.2), at the expense of a multiplicative factor of $1/4$. This leads precisely to the quadruple sum given in the statement of the Lemma. $\square$

3. Four double sums, with integral representations

A simple partial fraction decomposition of part of the summand in (2.2) affords a great deal of simplification in the expression (2.2) of the boundary-influenced correlation.

Indeed, one readily checks that

$$
\begin{aligned}
&\frac{(b-a)(d-c)}{(u+a+c)(u+a+d)(u+b+c)(u+b+d)}\\
&\qquad=\frac{1}{(u+a+c)(u+b+d)}-\frac{1}{(u+a+d)(u+b+c)}.
\end{aligned}
$$

Using this, the portion of the summand contained in the last line of (2.2) becomes

$$
\begin{aligned}
&\frac{(b-a)(d-c)}{(u+a+c)(u+b+d)}-\frac{(b-a)(d-c)}{(u+a+d)(u+b+c)}\\
&\qquad=\frac{ac+bd-ad-bc}{(u+a+c)(u+b+d)}-\frac{ac+bd-ad-bc}{(u+a+d)(u+b+c)}.
\end{aligned}
\tag{3.1}
$$

Using this, the fourfold sum of (2.2) becomes a difference of two fourfold sums. The advantage of this expression is that each of the latter two fourfold sums can be written as the product of two double sums. Indeed, in all the factors of the summand in (2.2) except the last line of (2.2), the summation variables can be separated. Furthermore, the variables $\{a, c\}$ can be separated from $\{b, d\}$ in the first term on the right hand side of (3.1), while $\{a, d\}$ can be separated from $\{b, c\}$ in the second term of the right hand side of (3.1). The double sums arising this way are all of the form

$$M_\nu(R_1, R_2) := \sum_{a=0}^{R_1} \sum_{c=0}^{R_2} (-1)^{a+c} \frac{(R_1 + a - 1)!}{(2a)!\,(R_1 - a)!} \frac{(R_2 + c - 1)!}{(2c + 1)!\,(R_2 - c)!} \\ \times \frac{2v_1 + 2a + 1)!}{2^{2v_1+2a}(v_1 + a)!\,(v_1 + a + 1)!} \frac{2v_2 + 2c + 1)!}{2^{2v_2+2c}(v_2 + c)!^2} \frac{\nu}{u + a + c}, \tag{3.2}$$

where ν has one of the values 1, a, c and ac. For notational convenience we will often write simply M_ν instead of $M_\nu(R_1, R_2)$.

More precisely, consider the term ac of the numerator of the first fraction on the right hand side of (3.1). When summing over a, b, c and d as required by (2.2), this term gives rise to $M_{ac}M_1$. Similarly, the remaining terms of that numerator, bd, $-ad$ and $-bc$, give rise to M_1M_{ac}, $-M_aM_c$ and $-M_aM_c$, respectively. In the same fashion, the second term on the right hand side of (3.1) generates the products $-M_aM_c$, $-M_aM_c$, $M_{ac}M_1$ and M_1M_{ac}, respectively. Therefore, we obtain by (2.2) that

$$\omega_b(R_1, v_1; R_2, v_2) = 2^{-4}R_1R_2(R_2 - 1/2)(R_2 + 1/2)\{4M_1M_{ac} - 4M_aM_c\} \\ = 2^{-2}R_1R_2(R_2 - 1/2)(R_2 + 1/2)(M_1M_{ac} - M_aM_c). \tag{3.3}$$

Thus, the asymptotic study of the correlation reduces to studying the asymptotics of these four double sums. By their definition (3.2), had it not been for the factor $1/(u + a + c)$, these double sums would further be separable as products of simple sums. We can get around the obstacle posed by this factor by expressing it as an integral[4]:

$$\frac{1}{u + a + c} = \int_0^1 x^{u+a+c-1} dx. \tag{3.4}$$

Indeed, substituting this into (3.2), we obtain that for instance M_1 is expressed as

$$M_1 = \int_0^1 \left(\sum_{a=0}^{R_1} (-1)^a \frac{(R_1 + a - 1)!}{(2a)!\,(R_1 - a)!} \frac{(2v_1 + 2a + 1)!}{2^{2v_1+2a}(v_1 + a)!\,(v_1 + a + 1)!} x^a \right) \\ \times \left(\sum_{c=0}^{R_2} (-1)^c \frac{(R_2 + c - 1)!}{(2c + 1)!\,(R_2 - c)!} \frac{(2v_2 + 2c + 1)!}{2^{2v_2+2c}(v_2 + c)!^2} x^c \right) x^{u-1} dx. \tag{3.5}$$

[4]This useful trick was pointed out to the author independently by Ira Gessel and Doron Zeilberger. Its use allows a shorter proof for the asymptotics of the four double sums than our original proof, which relied on the expansion

$$\frac{1}{u + a + b} = \frac{1}{u} \left\{ 1 - \frac{a + c}{u} + \frac{(a + c)(a + c - 1)}{u(u + 1)} - \frac{(a + c)(a + c - 1)(a + c - 2)}{u(u + 1)(u + 2)} + \cdots \right\}.$$

It is not difficult (indeed, with access to a computer algebra package like Maple, it is immediate) to see that the two sums in the above integral can be expressed in terms of hypergeometric functions[5] as

$$\sum_{a=0}^{R_1}(-1)^a \frac{(R_1+a-1)!}{(2a)!\,(R_1-a)!}\frac{(2v_1+2a+1)!}{2^{2v_1+2a}(v_1+a)!\,(v_1+a+1)!}x^a$$
$$= \frac{1}{R_1}\frac{(2v_1+1)!}{2^{2v_1}v_1!\,(v_1+1)!}\,{}_3F_2\left[\begin{matrix}-R_1, R_1, v_1+\frac{3}{2}\\ \frac{1}{2}, v_1+2\end{matrix}; \frac{x}{4}\right] \tag{3.6}$$

and

$$\sum_{c=0}^{R_2}(-1)^c \frac{(R_2+c-1)!}{(2c+1)!\,(R_2-c)!}\frac{(2v_2+2c+1)!}{2^{2v_2+2c}(v_1+c)!^2}x^c$$
$$= \frac{1}{R_2}\frac{(2v_2+1)!}{2^{2v_2}v_2!^2}\,{}_3F_2\left[\begin{matrix}-R_2, R_2, v_2+\frac{3}{2}\\ \frac{3}{2}, v_2+1\end{matrix}; \frac{x}{4}\right]. \tag{3.7}$$

Replacing these formulas in (3.5) we obtain the following result.

Proposition 3.1. *The double sum M_1 has the integral representation*

$$M_1 = \frac{1}{R_1R_2}\frac{(2v_1+1)!\,(2v_2+1)!}{2^{2v_1+2v_2}v_1!\,(v_1+1)!\,v_2!^2}$$
$$\times\int_0^1 {}_3F_2\left[\begin{matrix}-R_1, R_1, v_1+\frac{3}{2}\\ \frac{1}{2}, v_1+2\end{matrix}; \frac{x}{4}\right]{}_3F_2\left[\begin{matrix}-R_2, R_2, v_2+\frac{3}{2}\\ \frac{3}{2}, v_2+1\end{matrix}; \frac{x}{4}\right]x^{u-1}dx. \tag{3.8}$$

When applying the same reasoning to the remaining double sums M_a, M_c and M_{ac}, two more sums besides (3.6) and (3.7) need to be expressed in terms of hypergeometric functions. These are

$$\sum_{a=0}^{R_1}(-1)^a \frac{(R_1+a-1)!}{(2a)!\,(R_1-a)!}\frac{2v_1+2a+1)!}{2^{2v_1+2a}(v_1+a)!\,(v_1+a+1)!}x^a a$$
$$= -R_1x\frac{(2v_1+3)(2v_1+1)!}{2^{2v_1+2}v_1!\,(v_1+2)!}\,{}_3F_2\left[\begin{matrix}-R_1+1, R_1+1, v_1+\frac{5}{2}\\ \frac{3}{2}, v_1+3\end{matrix}; \frac{x}{4}\right] \tag{3.9}$$

[5]The hypergeometric function of parameters $a_1,\dots,a_p$ and $b_1,\dots,b_q$ is defined by

$${}_pF_q\left[\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}; z\right] = \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{k!\,(b_1)_k\cdots(b_q)_k}z^k\ ,$$

where $(a)_0 := 1$ and $(a)_k := a(a+1)\cdots(a+k-1)$ for $k\ge 1$.

and

$$\sum_{c=0}^{R_2}(-1)^c \frac{(R_2+c-1)!}{(2c+1)!\,(R_2-c)!}\frac{2v_2+2c+1)!}{2^{2v_2+2c}(v_1+c)!^2}x^c c$$
$$= -\frac{R_2 x}{3}\frac{(2v_2+3)(2v_2+1)!}{2^{2v_2+2}v_2!\,(v_2+1)!}\,{}_3F_2\left[\begin{matrix}-R_2+1, R_2+1, v_2+\frac{5}{2}\\ \frac{5}{2}, v_2+2\end{matrix}; \frac{x}{4}\right]. \tag{3.10}$$

Using (3.2), (3.4), (3.6), (3.7), (3.9) and (3.10), we obtain the following result.

PROPOSITION 3.2. *The double sums M_a, M_c and M_{ac} have the integral representations*

$$M_a = -\frac{R_1}{R_2}\frac{(2v_1+3)(2v_1+1)!\,(2v_2+1)!}{2^{2v_1+2v_2+2}v_1!\,(v_1+2)!\,v_2!^2}$$
$$\times \int_0^1 {}_3F_2\left[\begin{matrix}-R_1+1, R_1+1, v_1+\frac{5}{2}\\ \frac{3}{2}, v_1+3\end{matrix}; \frac{x}{4}\right]\,{}_3F_2\left[\begin{matrix}-R_2, R_2, v_2+\frac{3}{2}\\ \frac{3}{2}, v_2+1\end{matrix}; \frac{x}{4}\right] x^u dx \tag{3.11}$$

$$M_c = -\frac{R_2}{3R_1}\frac{(2v_1+1)!\,(2v_2+1)!\,(2v_2+3)}{2^{2v_1+2v_2+2}v_1!\,(v_1+1)!\,v_2!\,(v_2+1)!}$$
$$\times \int_0^1 {}_3F_2\left[\begin{matrix}-R_1, R_1, v_1+\frac{3}{2}\\ \frac{1}{2}, v_1+2\end{matrix}; \frac{x}{4}\right]\,{}_3F_2\left[\begin{matrix}-R_2+1, R_2+1, v_2+\frac{5}{2}\\ \frac{5}{2}, v_2+2\end{matrix}; \frac{x}{4}\right] x^u dx \tag{3.12}$$

$$M_{ac} = \frac{R_1R_2}{3}\frac{(2v_1+3)(2v_1+1)!\,(2v_2+3)(2v_2+1)!}{2^{2v_1+2v_2+4}v_1!\,(v_1+2)!\,v_2!\,(v_2+1)!}$$
$$\times \int_0^1 {}_3F_2\left[\begin{matrix}-R_1+1, R_1+1, v_1+\frac{5}{2}\\ \frac{3}{2}, v_1+3\end{matrix}; \frac{x}{4}\right]\,{}_3F_2\left[\begin{matrix}-R_2+1, R_2+1, v_2+\frac{5}{2}\\ \frac{5}{2}, v_2+2\end{matrix}; \frac{x}{4}\right] x^{u+1} dx. \tag{3.13}$$

4. Jacobi polynomials

The ${}_3F_2$'s of the preceding section can be expressed in terms of ${}_2F_1$'s by formula (2) on page 497 of [**PBM**]

$${}_3F_2\left[\begin{matrix}a,\, b,\, c\\ a-n, d\end{matrix}; z\right] = \frac{1}{(1-a)_n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1-a)_{n-k}\frac{(b)_k(c)_k}{(d)_k}z^k\,{}_2F_1\left[\begin{matrix}b+k, c+k\\ d+k\end{matrix}; z\right], \tag{4.1}$$

where n is a nonnegative integer. Applying this to the ${}_3F_2$'s in (3.6) and (3.7), one obtains

$${}_3F_2\left[\begin{matrix}-R_1, R_1, v_1+\frac{3}{2}\\ \frac{1}{2}, v_1+2\end{matrix}; \frac{x}{4}\right] = \frac{1}{(-v_1-\frac{1}{2})_{v_1+1}}$$
$$\times \sum_{k=0}^{v_1+1}(-1)^k\binom{v_1+1}{k}(-v_1-\frac{1}{2})_{v_1+1-k}\frac{(-R_1)_k(R_1)_k}{(v_1+2)_k}\frac{x^k}{4^k}\,{}_2F_1\left[\begin{matrix}-R_1+k, R_1+k\\ v_1+2+k\end{matrix}; \frac{x}{4}\right] \tag{4.2}$$

and

$$
{}_3F_2\left[\begin{matrix}-R_2, R_2, v_2+\frac{3}{2} \\ \frac{3}{2}, v_2+1\end{matrix}; \frac{x}{4}\right] = \frac{1}{(-v_2-\frac{1}{2})_{v_2}}
$$

$$
\times \sum_{l=0}^{v_2} (-1)^l \binom{v_2}{l} (-v_2-\frac{1}{2})_{v_2-l} \frac{(-R_2)_l (R_2)_l}{(v_2+1)_l} \frac{x^l}{4^l} {}_2F_1\left[\begin{matrix}-R_2+l, R_2+l \\ v_2+1+l\end{matrix}; \frac{x}{4}\right]. \tag{4.3}
$$

In turn, the resulting ${}_2F_1$'s can be expressed in terms of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ using the formula (found for instance in 8.962, page 1060 of [**GR**])

$$
{}_2F_1\left[\begin{matrix}n+\alpha+\beta+1, -n \\ 1+\alpha\end{matrix}; \frac{1-x}{2}\right] = \frac{n!\,\Gamma(1+\alpha)}{\Gamma(n+1+\alpha)} P_n^{(\alpha,\beta)}(x). \tag{4.4}
$$

We obtain for the ${}_2F_1$'s of (4.2) and (4.3) the expressions

$$
{}_2F_1\left[\begin{matrix}-R_1+k, R_1+k \\ v_1+2+k\end{matrix}; \frac{x}{4}\right] = \frac{(R_1-k)!\,(v_1+k+1)!}{(R_1+v_1+1)!} P_{R_1-k}^{(v_1+k+1,k-v_1-2)}\left(1-\frac{x}{2}\right) \tag{4.5}
$$

and

$$
{}_2F_1\left[\begin{matrix}-R_2+l, R_2+l \\ v_2+1+l\end{matrix}; \frac{x}{4}\right] = \frac{(R_2-l)!\,(v_2+l)!}{(R_2+v_2)!} P_{R_2-l}^{(v_2+l,l-v_2-1)}\left(1-\frac{x}{2}\right). \tag{4.6}
$$

Substituting (4.2), (4.3), (4.5) and (4.6) in the integral representation (3.8) of M_1, we obtain that

$$
M_1 = \sum_{k=0}^{v_1+1} \sum_{l=0}^{v_2} \frac{c_{kl}}{R_1 R_2} \frac{(-R_1)_k (R_1)_k}{(R_1-k+1)_{v_1+k+1}} \frac{(-R_2)_l (R_2)_l}{(R_2-l+1)_{v_2+l}} I_{kl}(R_1,R_2), \tag{4.7}
$$

where c_{kl} depends only on k, l, v_1 and v_2, and for $0 \le k \le v_1+1$ and $0 \le l \le v_2$,

$$
I_{kl}(R_1,R_2) := \int_0^1 P_{R_1-k}^{(v_1+k+1,k-v_1-2)}\left(1-\frac{x}{2}\right) P_{R_2-l}^{(v_2+l,l-v_2-1)}\left(1-\frac{x}{2}\right) x^{k+l+u-1}\,dx. \tag{4.8}
$$

By (1.4) and (3.3), we need the asymptotics of M_1 for $v_1 = u-1$, $v_2 = 0$, $R_1 = R+r$ and $R_2 = R$, where $r \ge 0$ and $u \ge 1$ are fixed and $R \to \infty$.

By (4.7), the study of this asymptotics of M_1 reduces to the asymptotics of the I_{kl}'s. The asymptotics of the Jacobi polynomials $P_n^{(\alpha,\beta)}$ for large n is given by the Darboux formula (see e.g. [**GR**, p.1061])

$$
\begin{aligned}
P_n^{(\alpha,\beta)}(\cos\theta) &= D_n^{(\alpha,\beta)}(\cos\theta) + E_n^{(\alpha,\beta)}(\cos\theta) \\
D_n^{(\alpha,\beta)}(\cos\theta) &= \frac{\cos\left\{\left[n+\frac{\alpha+\beta+1}{2}\right]\theta - \left(\frac{\alpha}{2}+\frac{1}{4}\right)\pi\right\}}{\sqrt{\pi n}\left(\sin\frac{\theta}{2}\right)^{\alpha+\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}} \\
E_n^{(\alpha,\beta)}(\cos\theta) &= O(n^{-3/2}),
\end{aligned} \tag{4.9}
$$

where $\alpha, \beta \in \mathbb{R}$ and $0 < \theta < \pi$ are fixed.

We show in Lemma 4.2 below that replacing in I_{kl} the Jacobi polynomials by their Darboux approximations leaves the asymptotics of I_{kl} unchanged. This will follow from the following inequalities due to Szegő [**Sz1**] (see also [**Sz2**, p.197]).

Proposition 4.1 (Szegő [**Sz1**, (46'), (48'), p.77][**Sz2**]). (*a*) *Let* $\alpha \geq -1/2$, $\beta \in \mathbb{R}$ *and* $\epsilon > 0$. *Then there exists a constant* A *depending only on* α, β *and* ϵ *so that*

$$\left|P_n^{(\alpha,\beta)}(x)\right| \leq \frac{A}{\sqrt{n}} \frac{1}{(1-x)^{\frac{\alpha}{2}+\frac{1}{4}}}, \qquad -1+\epsilon \leq x \leq 1. \tag{4.10}$$

(*b*) *Let* $\alpha, \beta \in \mathbb{R}$ *and* $\epsilon, c > 0$. *Then there exists a constant* B *depending only on* α, β, ϵ *and* c *so that*

$$\left|E_n^{(\alpha,\beta)}(\cos\theta)\right| \leq \frac{B}{n^{3/2}} \frac{1}{\theta^{\alpha+\frac{3}{2}}}, \qquad cn^{-1} \leq \theta \leq \pi - \epsilon. \tag{4.11}$$

One readily checks that

$$\cos^{-1}(x) \geq \sqrt{2}\sqrt{1-x}, \qquad 0 \leq x \leq 1. \tag{4.12}$$

Indeed, if $f(x) := \cos^{-1}(x) - \sqrt{2}\sqrt{1-x}$, one has $f'(x) = (1-x)^{-1/2}(2^{-1/2} - (1+x)^{-1/2}) < 0$, for $x \in [0,1)$, and as $f(1) = 0$, (4.12) follows.

Using (4.12) and Proposition 4.1(b) one obtains that for any $\epsilon > 0$ there exists constants B_ϵ and N_ϵ depending only on α, β and ϵ so that

$$\left|E_n^{(\alpha,\beta)}(x)\right| \leq \frac{B_\epsilon}{n^{3/2}} \frac{1}{(1-x)^{\frac{\alpha}{2}+\frac{3}{4}}}, \qquad \frac{1}{2} \leq x \leq 1-\epsilon,\ n \geq N_\epsilon. \tag{4.13}$$

By Proposition 4.1(a), there exists a constant A depending only on α and β so that

$$\left|P_n^{(\alpha,\beta)}(x)\right| \leq \frac{A}{\sqrt{n}} \frac{1}{(1-x)^{\frac{\alpha}{2}+\frac{1}{4}}}, \qquad \frac{1}{2} \leq x \leq 1. \tag{4.14}$$

Using (4.9) and the fact that $\sin(\cos^{-1}(x)/2) = \sqrt{(1-x)/2}$ and $\cos(\cos^{-1}(x)/2) = \sqrt{(1+x)/2}$, one obtains that the Darboux approximants satisfy an inequality of the same form as (4.14): there exists a constant A' such that

$$\left|D_n^{(\alpha,\beta)}(x)\right| \leq \frac{A'}{\sqrt{n}} \frac{1}{(1-x)^{\frac{\alpha}{2}+\frac{1}{4}}}, \qquad \frac{1}{2} \leq x \leq 1. \tag{4.15}$$

By (4.14) and (4.15), $E_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x) - D_n^{(\alpha,\beta)}(x)$ also satisfies an inequality of the same type, so there exists a constant A'' depending only on α and β so that

$$\left|E_n^{(\alpha,\beta)}(x)\right| \leq \frac{A''}{\sqrt{n}} \frac{1}{(1-x)^{\frac{\alpha}{2}+\frac{1}{4}}}, \qquad \frac{1}{2} \leq x \leq 1. \tag{4.16}$$

We are now ready to prove the announced invariance of the asymptotics of $I_{kl}(R_1, R_2)$ under replacement of the Jacobi polynomials by their Darboux approximants. Define

$$J_{kl}(R_1,R_2) := \int_0^1 D_{R_1-k}^{(v_1+k+1,k-v_1-2)}\left(1-\frac{x}{2}\right) D_{R_2-l}^{(v_2+l,l-v_2-1)}\left(1-\frac{x}{2}\right) x^{k+l+u-1} dx. \tag{4.17}$$

LEMMA 4.2. *Let r, v_1, v_2, $0 \le k \le v_1+1$ and $0 \le l \le v_2$ be fixed, and assume that not all of v_1, v_2, k and l are zero. Then we have*

$$\lim_{R\to\infty} R\left(I_{kl}(R+r,R) - J_{kl}(R+r,R)\right) = 0.$$

Proof. Denoting for simplicity $f(x) := P_{R_1-k}^{(v_1+k+1,k-v_1-2)}(x)$, $g(x) := P_{R_2-l}^{(v_2+l,l-v_2-1)}(x)$, $F(x) := D_{R_1-k}^{(v_1+k+1,k-v_1-2)}(x)$ and $G(x) := D_{R_2-l}^{(v_2+l,l-v_2-1)}(x)$, we obtain

$$\begin{aligned}
&|I_{kl}(R_1,R_2) - J_{kl}(R_1,R_2)| \\
&= \left|\int_0^1 \{f(1-x/2)g(1-x/2) - F(1-x/2)G(1-x/2)\}\, x^{k+l+u-1}dx\right| \\
&= |\int_0^1 \{f(1-x/2)g(1-x/2) - f(1-x/2)G(1-x/2) + f(1-x/2)G(1-x/2) \\
&\qquad\qquad - F(1-x/2)G(1-x/2)\}x^{k+l+u-1}dx| \\
&\le \int_0^1 |f(1-x/2)|\,|g(1-x/2) - G(1-x/2)|\, x^{k+l+u-1}dx \\
&\qquad + \int_0^1 |G(1-x/2)|\,|f(1-x/2) - F(1-x/2)G(1-x/2)|\, x^{k+l+u-1}dx.
\end{aligned}$$

Therefore, to prove the Lemma it suffices to show that for $R_1 = R+r$ and $R_2 = R$,

$$\lim_{R\to\infty} R\int_0^1 \left|P_{R_1-k}^{(v_1+k+1,k-v_1-2)}(1-x/2)\right| \left|E_{R_2-l}^{(v_2+l,l-v_2-1)}(1-x/2)\right| x^{k+l+u-1}dx = 0 \tag{4.18}$$

and

$$\lim_{R\to\infty} R\int_0^1 \left|E_{R_1-k}^{(v_1+k+1,k-v_1-2)}(1-x/2)\right| \left|D_{R_2-l}^{(v_2+l,l-v_2-1)}(1-x/2)\right| x^{k+l+u-1}dx = 0. \tag{4.19}$$

By the Cauchy-Schwarz inequality, for h with integrable square we have

$$\left(\int_0^1 h(x)dx\right)^2 \le \int_0^1 h^2(x)dx. \tag{4.20}$$

Let $0 < \epsilon < 1$ be arbitrary. By (4.20) we have

$$\begin{aligned}
&\left(\int_0^1 \left|P_{R_1-k}^{(v_1+k+1,k-v_1-2)}(1-x/2)\right| \left|E_{R_2-l}^{(v_2+l,l-v_2-1)}(1-x/2)\right| x^{k+l+u-1}dx\right)^2 \\
&\le \int_0^{1-\epsilon} \left|P_{R_1-k}^{(v_1+k+1,k-v_1-2)}(1-x/2)\right|^2 \left|E_{R_2-l}^{(v_2+l,l-v_2-1)}(1-x/2)\right|^2 x^{2(k+l+u-1)}dx \\
&+ \int_{1-\epsilon}^1 \left|P_{R_1-k}^{(v_1+k+1,k-v_1-2)}(1-x/2)\right|^2 \left|E_{R_2-l}^{(v_2+l,l-v_2-1)}(1-x/2)\right|^2 x^{2(k+l+u-1)}dx.
\end{aligned} \tag{4.21}$$

Denote the two integrals on the right hand side of (4.21) by I_1 and I_2, respectively. By (4.13) and (4.14), we have

$$I_1 \le \frac{A^2 B_\epsilon^2}{(R_1-k)(R_2-l)^3}\int_0^{1-\epsilon}(x/2)^{-(v_1+k+1+\frac{1}{2})}(x/2)^{-(v_2+l+\frac{3}{2})}x^{2(k+l+u-1)}dx$$
$$\le \frac{M_\epsilon}{R_1R_2^3}\int_0^{1-\epsilon}x^{v_1+v_2+k+l-1}dx, \tag{4.22}$$

where M_ϵ depends on v_1, v_2, k, l and ϵ, but is independent of R_1 and R_2 (here we used that $u = v_1+v_2+2$). Since by hypothesis v_1, v_2, k and l are not all equal to zero, the exponent of the integrand in (4.22) is nonnegative and we obtain

$$I_1 \le \frac{M_\epsilon}{R_1R_2^3}. \tag{4.23}$$

On the other hand, by (4.14) and (4.16), we have that

$$I_2 \le \frac{AA''}{(R_1-k)(R_2-l)}\int_{1-\epsilon}^{1}(x/2)^{-(v_1+k+1+\frac{1}{2})}(x/2)^{-(v_2+l+\frac{1}{2})}x^{2(k+l+u-1)}dx$$
$$\le \frac{M'}{R_1R_2}\int_{1-\epsilon}^{1}x^{v_1+v_2+k+l}dx$$
$$\le \frac{\epsilon M'}{R_1R_2}, \tag{4.24}$$

where M' is independent of ϵ, R_1 and R_2, depending just on v_1, v_2, k and l.

By (4.21), (4.23) and (4.24), for $R_1 = R+r$ and $R_2 = R$ the first term in (4.21) is majorized by

$$I_1 + I_2 \le \frac{\epsilon M'}{R^2} + \frac{M_\epsilon}{R^4} \le \frac{1}{R^2}\left(\sqrt{\epsilon M'} + \frac{\sqrt{M_\epsilon}}{R}\right)^2.$$

Extracting the square root we obtain

$$\int_0^1 \left|P_{R_1-k}^{(v_1+k+1,k-v_1-2)}(1-x/2)\right|\left|E_{R_2-l}^{(v_2+l,l-v_2-1)}(1-x/2)\right|x^{k+l+u-1}dx$$
$$\le \frac{\sqrt{\epsilon M'}}{R} + \frac{\sqrt{M_\epsilon}}{R^2}.$$

Multiplying the previous inequality by R, we obtain that the quantity whose limit is taken in (4.18) is majorized by $\sqrt{\epsilon M'} + \sqrt{M_\epsilon}/R$. This quantity can be made arbitrarirly small by first choosing ϵ so as to make $\sqrt{\epsilon M'}$ arbitrarily small, and then requiring R to be large enough to make $\sqrt{M_\epsilon}/R$ arbitrarily small. This proves (4.18).

A similar argument proves (4.19). This completes the proof of the Lemma. □

5. The asymptotics of M_1

The following result will be needed several times during the remaining part of the paper.

LEMMA 5.1. *Let $\alpha(t)$, $k(t)$ and $h(t)$ be complex-valued functions that are real for real t and analytic in a domain $\mathbf{T} \subset \mathbb{C}$ containing the interval $(0,1]$. Let $q > 0$ be fixed. Then*

$$\int_0^1 t^{Rq} h(t) \cos[R\alpha(t) + k(t)] dt$$
$$= \frac{h(1)}{R\sqrt{q^2 + (\alpha'(1))^2}} \cos\left[R\alpha(1) + k(1) - \arctan\frac{\alpha'(1)}{q}\right] + O(R^{-2}). \tag{5.1}$$

Proof. We have

$$\int_0^1 t^{Rq} h(t) \cos[R\alpha(t) + k(t)] dt$$
$$= \frac{1}{2}\int_0^1 e^{Rq\ln t}\left[e^{i(R\alpha(t)+k(t))} + e^{-i(R\alpha(t)+k(t))}\right] h(t) dt$$
$$= -\frac{1}{2}\left\{\int_1^0 e^{-R[-q\ln t - i\alpha(t)]} e^{ik(t)} h(t) dt + \int_1^0 e^{-R[-q\ln t + i\alpha(t)]} e^{-ik(t)} h(t) dt\right\}. \tag{5.2}$$

The asymptotics for large R of each of the two integrals on the last line of (5.2) can be found by the Laplace method as it is described for example in §6, Chapter 4 of [**O**]. Indeed, consider the first integral. The only requierment of the hypothesis of Theorem 6.1 of [**O**, p. 125] that needs to be checked is that the real part of the coefficient of $-R$ in the first exponential in the integrand attains its minimum at $t = 1$. This indeed holds, since $-\ln t$ has its minimum at $t = 1$. The relevant quantities are easily found to be, in the notation of the quoted theorem of [**O**], $\lambda = \mu = 1$, $p_0 = -q - i\alpha'(1)$, $q_0 = e^{ik(1)}h(1)$ and $p(1) = -i\alpha(1)$. By that theorem, the value of the integral is $a_0 e^{-Rp(1)}/R + O(R^{-2})$, where $a_0 = q_0/(\mu p_0^{\lambda/\mu})$. Therefore, we obtain by Theorem 6.1 of [**O**, p. 125] that

$$\int_1^0 e^{-R[-q\ln t - i\alpha(t)]} e^{ik(t)} h(t) dt = -\frac{e^{i(R\alpha(1)+k(1))} h(1)}{q + i\alpha'(1)} \frac{1}{R} + O(R^{-2}). \tag{5.3}$$

The two integrals on the last line of (5.2) are complex conjugates, so the coefficients of their asymptotic expansions are also complex conjugates. We obtain from (5.3) that

$$\int_1^0 e^{-R[-q\ln t + i\alpha(t)]} e^{-ik(t)} h(t) dt = -\frac{e^{-i(R\alpha(1)+k(1))} h(1)}{q - i\alpha'(1)} \frac{1}{R} + O(R^{-2}). \tag{5.4}$$

However, it is readily checked that

$$\frac{e^{i\varphi}}{q + ib} + \frac{e^{-i\varphi}}{q - ib} = \frac{2}{\sqrt{q^2 + b^2}} \cos(\varphi - \theta), \tag{5.5}$$

where $\theta = \arctan(b/q)$. By (5.2)–(5.5) we obtain the statement of the Lemma. □

LEMMA 5.2. *For fixed r, v_1 and v_2 we have*

$$I_{v_1+1,v_2}(R+r,R) = \frac{(-4)^{v_1+v_2+1}}{2R\pi}\int_0^1\left(\frac{4-x}{x}\right)^{1/2} x^{v_1+v_2+1}\cos\left[r\cos^{-1}\left(1-\frac{x}{2}\right)\right]dx + o(R^{-1}). \tag{5.6}$$

Proof. By Lemma 4.2, it is enough to show that $J_{v_1+1,v_2}(R+r,R)$ has the asymptotics given by (4.25). Using (4.17), (4.9) and the fact that $\sin(\cos^{-1}(1-x/2)/2) = (x/4)^{1/2}$, $\cos(\cos^{-1}(1-x/2)/2) = ((4-x)/4)^{1/2}$ and $\cos(z-n\pi) = (-1)^n\cos(z)$, one obtains that

$$\begin{aligned} &J_{v_1+1,v_2}(R+r,R) \\ &= \frac{(-4)^{v_1+v_2+1}}{\pi\sqrt{R+r-v_1-1}\sqrt{R-v_2}}\int_0^1\left(\frac{4-x}{x}\right)^{1/2} x^{v_1+v_2+1}\cos\left[(R+r)\cos^{-1}\left(1-\frac{x}{2}\right)-\frac{\pi}{4}\right] \\ &\qquad\times\cos\left[R\cos^{-1}\left(1-\frac{x}{2}\right)-\frac{\pi}{4}\right]dx. \end{aligned} \tag{5.7}$$

Converting the product of cosines into a sum, we obtain that

$$J_{v_1+1,v_2}(R+r,R) = \frac{(-4)^{v_1+v_2+1}}{2\pi\sqrt{R+r-v_1-1}\sqrt{R-v_2}}(J_1+J_2), \tag{5.8}$$

where

$$\begin{aligned} J_1 &= \int_0^1\left(\frac{4-x}{x}\right)^{1/2} x^{v_1+v_2+1}\cos\left[r\cos^{-1}\left(1-\frac{x}{2}\right)\right]dx, \\ J_2 &= \int_0^1\left(\frac{4-x}{x}\right)^{1/2} x^{v_1+v_2+1}\cos\left[(2R+r)\cos^{-1}\left(1-\frac{x}{2}\right)-\frac{\pi}{2}\right]dx. \end{aligned} \tag{5.9}$$

By Lemma 5.1, $J_2 = O(1/R)$. Therefore, (5.7)–(5.9) imply that the asymptotics of $J_{v_1+1,v_2}(R+r,R)$ is given by the right hand side of (5.6). As noted in the beginning of the proof, this completes the proof of the Lemma. □

We are now ready to give the asymptotics of M_1.

PROPOSITION 5.3. *For fixed r, v_1 and v_2, we have*

$$M_1(R+r,R) = \frac{1}{R^3}\frac{1}{\pi}\int_0^1\left(\frac{4-x}{x}\right)^{1/2} x^{u-1}\cos\left[r\cos^{-1}\left(1-\frac{x}{2}\right)\right]dx + o(R^{-3}), \tag{5.10}$$

where $u = v_1+v_2+2$.

Proof. Using the bounds (4.14) for the Jacobi polynomials, we obtain from (4.8) that

$$\begin{aligned} I_{kl}(R_1,R_2) &\le \frac{M}{\sqrt{R_1-k}\sqrt{R_2-l}}\int_0^1 \frac{1}{\left(\frac{x}{2}\right)^{\frac{v_1+k+1}{2}+\frac{1}{4}}}\frac{1}{\left(\frac{x}{2}\right)^{\frac{v_2+l}{2}+\frac{1}{4}}} x^{v_1+v_2+k+l+1}dx \\ &\le \frac{M'}{\sqrt{R_1}\sqrt{R_2}}\int_0^1 x^{\frac{v_1+v_2+k+l}{2}}dx, \end{aligned} \tag{5.11}$$

where the constants M and M' depend just on v_1, v_2, k and l. Therefore, we obtain that

$$I_{kl}(R+r,R) = O\left(\frac{1}{R}\right), \qquad 0 \le k \le v_1+1, 0 \le l \le v_2. \tag{5.12}$$

Consider now the representation of M_1 given by (4.7). For $R_1 = R+r$ and $R_2 = R$, the coefficient of each $I_{kl}(R_1,R_2)$ with $(k,l) \neq (v_1+1, v_2)$ is $O(R^{-3})$. Thus, by (5.12) we obtain from (4.7) that

$$M_1(R+r,R) = \frac{c_{v_1+1,v_2}}{R^2} I_{v_1+1,v_2}(R+r,R) + O\left(\frac{1}{R^4}\right). \tag{5.13}$$

By (3.8), (4.2), (4.3), (4.5) and (4.6) we obtain after simplifications that

$$c_{v_1+1,v_2} = \frac{2}{(-4)^{v_1+v_2+1}}.$$

Substituting this value into (5.13) and using Lemma 5.2 we obtain the statement of the Proposition. $\square$

6. The asymptotics of M_a, M_c and M_{ac}

Our analysis of the asymptotics of M_1 can be repeated for the remaining double sums M_a, M_c and M_{ac}. We obtain the following result.

PROPOSITION 6.1. *For fixed r, v_1 and v_2, we have*

$$M_a(R+r,R) = -\frac{1}{R^2}\frac{1}{\pi}\int_0^1 x^{u-1} \cos\left[(r-1)\cos^{-1}\left(1-\frac{x}{2}\right) - \frac{\pi}{2}\right] dx + o(R^{-2}) \tag{6.1}$$

$$M_c(R+r,R) = \frac{1}{R^2}\frac{1}{\pi}\int_0^1 x^{u-1} \cos\left[(r+1)\cos^{-1}\left(1-\frac{x}{2}\right) - \frac{\pi}{2}\right] dx + o(R^{-2}) \tag{6.2}$$

$$M_{ac}(R+r,R) = \frac{1}{R}\frac{1}{\pi}\int_0^1 \left(\frac{4-x}{x}\right)^{-1/2} x^{u-1} \cos\left[r\cos^{-1}\left(1-\frac{x}{2}\right)\right] dx + o(R^{-1}) \tag{6.3}$$

where $u = v_1+v_2+2$.

Proof. By (4.1) we can express the ${}_3F_2$'s of (3.9) and (3.10) as

$$\begin{aligned} {}_3F_2\left[\begin{matrix}-R_1+1, R_1+1, v_1+\frac{5}{2} \\ \frac{3}{2}, v_1+3\end{matrix}; \frac{x}{4}\right] &= \frac{1}{(-v_1-\frac{3}{2})_{v_1+1}} \sum_{k=0}^{v_1+1} (-1)^k \binom{v_1+1}{k} (-v_1-\frac{3}{2})_{v_1+1-k} \\ &\times \frac{(-R_1+1)_k (R_1+1)_k}{(v_1+3)_k} \frac{x^k}{4^k}\, {}_2F_1\left[\begin{matrix}-R_1+1+k, R_1+1+k \\ v_1+3+k\end{matrix}; \frac{x}{4}\right] \end{aligned} \tag{6.4}$$

and

$$
{}_3F_2\left[\begin{matrix}-R_2+1,R_2+1,v_2+\frac{5}{2}\\ \frac{5}{2},v_2+2\end{matrix};\frac{x}{4}\right]=\frac{1}{(-v_2-\frac{3}{2})_{v_2}}\sum_{l=0}^{v_2}(-1)^l\binom{v_2}{l}(-v_2-\frac{3}{2})_{v_2-l}
$$

$$
\times\frac{(-R_2+1)_l(R_2+1)_l}{(v_2+2)_l}\frac{x^l}{4^l}\,{}_2F_1\left[\begin{matrix}-R_2+1+l,R_2+1+l\\ v_2+2+l\end{matrix};\frac{x}{4}\right]. \tag{6.5}
$$

By (4.4), the resulting ${}_2F_1$'s are expressed in terms of Jacobi polynomials as

$$
{}_2F_1\left[\begin{matrix}-R_1+1+k,R_1+1+k\\ v_1+3+k\end{matrix};\frac{x}{4}\right]=\frac{(R_1-k-1)!\,(v_1+k+2)!}{(R_1+v_1+1)!}P_{R_1-k-1}^{(v_1+k+2,k-v_1-1)}\left(1-\frac{x}{2}\right) \tag{6.6}
$$

and

$$
{}_2F_1\left[\begin{matrix}-R_2+1+l,R_2+1+l\\ v_2+2+l\end{matrix};\frac{x}{4}\right]=\frac{(R_2-l-1)!\,(v_2+l+1)!}{(R_2+v_2)!}P_{R_2-l-1}^{(v_2+l+1,l-v_2)}\left(1-\frac{x}{2}\right). \tag{6.7}
$$

Substituting the expansions (6.4) and (4.3) and the formulas (6.6) and (4.6) into the integral representation (3.11) of M_a, we obtain that

$$
M_a=\sum_{k=0}^{v_1+1}\sum_{l=0}^{v_2}\frac{c'_{kl}R_1}{R_2}\frac{(-R_1+1)_k(R_1+1)_k}{(R_1-k)_{v_1+k+2}}\frac{(-R_2)_l(R_2)_l}{(R_2-l+1)_{v_2+l}}I'_{kl}(R_1,R_2), \tag{6.8}
$$

where c'_{kl} depends only on k, l, v_1 and v_2, and for $0\le k\le v_1+1$ and $0\le l\le v_2$,

$$
I'_{kl}(R_1,R_2):=\int_0^1 P_{R_1-k-1}^{(v_1+k+2,k-v_1-1)}\left(1-\frac{x}{2}\right)P_{R_2-l}^{(v_2+l,l-v_2-1)}\left(1-\frac{x}{2}\right)x^{k+l+u}dx. \tag{6.9}
$$

Similarly, we get from (3.12) that

$$
M_c=\sum_{k=0}^{v_1+1}\sum_{l=0}^{v_2}\frac{c''_{kl}R_2}{R_1}\frac{(-R_1)_k(R_1)_k}{(R_1-k+1)_{v_1+k+1}}\frac{(-R_2+1)_l(R_2+1)_l}{(R_2-l)_{v_2+l+1}}I''_{kl}(R_1,R_2), \tag{6.10}
$$

where c''_{kl} depends only on k, l, v_1 and v_2, and for $0\le k\le v_1+1$ and $0\le l\le v_2$,

$$
I''_{kl}(R_1,R_2):=\int_0^1 P_{R_1-k}^{(v_1+k+1,k-v_1-2)}\left(1-\frac{x}{2}\right)P_{R_2-l-1}^{(v_2+l+1,l-v_2)}\left(1-\frac{x}{2}\right)x^{k+l+u}dx. \tag{6.11}
$$

An analogous calculation yields from (3.13) that

$$
M_{ac}=\sum_{k=0}^{v_1+1}\sum_{l=0}^{v_2}c'''_{kl}R_2R_1\frac{(-R_1+1)_k(R_1+1)_k}{(R_1-k)_{v_1+k+2}}\frac{(-R_2+1)_l(R_2+1)_l}{(R_2-l)_{v_2+l+1}}I'''_{kl}(R_1,R_2), \tag{6.12}
$$

where c'''_{kl} depends only on k, l, v_1 and v_2, and for $0 \le k \le v_1+1$ and $0 \le l \le v_2$,

$$I'''_{kl}(R_1,R_2) := \int_0^1 P_{R_1-k-1}^{(v_1+k+2,k-v_1-1)}\left(1-\frac{x}{2}\right) P_{R_2-l-1}^{(v_2+l+1,l-v_2)}\left(1-\frac{x}{2}\right) x^{k+l+u+1} dx. \quad (6.13)$$

As seen in the proof of Lemma 5.2 for the case of I_{kl}, the bounds (4.14) imply that for fixed r, v_1, v_2 and fixed $0 \le k \le v_1+1$ and $0 \le l \le v_2$, the integrals $I'_{kl}(R+r,R)$, $I''_{kl}(R+r,R)$ and $I'''_{kl}(R+r,R)$ are $O(1/R)$. Indeed, the only change from that case is that now the α-parameters of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ that occur are slightly changed. The key fact needed to prove (5.12) was that the exponent of x in the last integral of (5.11) was nonnegative. However, the analogous exponents for the case of $I'_{k,l}$, $I''_{k,l}$ and $I'''_{k,l}$ are readily seen to be nonnegative as well (by (4.14), this exponent goes down half a unit for each unit of increase in the α-parameter of the Jacobi polynomials that occur; the α-parameters of the pairs of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ appearing in I'_{kl}, I''_{kl} and I'''_{kl} are increased by $(1,0)$, $(0,1)$ and $(1,1)$, respectively; the increase in the exponent of x in (3.11)–(3.13), namely 1, 1, and 2 units, respectively, makes up for the decrease due to the change in the α-parameters).

Using this, it can be shown that, just as it was the case for the expansion (4.7) of M_1, the asymptotics of the double sums (6.8), (6.10) and (6.12) for r, v_1, v_2 fixed and $R_1 = R+r$, $R_2 = R$, $R \to \infty$ are given by the contribution of the terms with $(k,l) = (v_1+1, v_2)$.

To see this, note first that analogs of Lemma 4.2 hold for I'_{v_1+1,v_2}, I''_{v_1+1,v_2} and I'''_{v_1+1,v_2}, with J'_{v_1+1,v_2}, J''_{v_1+1,v_2} and J'''_{v_1+1,v_2} defined by replacing the Jacobi polynomials in the integrands of the I-integrals by their Darboux approximants. Indeed, the only difference from the calculations in the proof of Lemma 4.2 is that now the α-parameters of the pairs of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ appearing in the I-integrals are increased by $(1,0)$, $(0,1)$ and $(1,1)$, respectively. However, just as was the case in the previous paragraph, the increase in the exponent of x in (6.9), (6.11) and (6.13) (of 1, 1 and 2 units, respectively) compensates the decrease due to the change in the α-parameters. Therefore, arguments parallel to the ones in the proof of Lemma 4.2 lead to analogs of the majorizations (4.22) and (4.24) that maintain the key feature of having non-negative exponents of x in the integrand, and thus prove the claimed analogs of Lemma 4.2.

Second, using these analogs of Lemma 4.2, one can easily deduce analogs of Lemma 5.2, yielding

$$I_{v_1+1,v_2}(R+r,R)$$
$$= -\frac{(-4)^{v_1+v_2+2}}{2R\pi}\int_0^1 x^{v_1+v_2+1}\cos\left[(r-1)\cos^{-1}\left(1-\frac{x}{2}\right)\right]dx + o(R^{-1}) \tag{6.14}$$

$$I_{v_1+1,v_2}(R+r,R)$$
$$= \frac{(-4)^{v_1+v_2+2}}{2R\pi}\int_0^1 x^{v_1+v_2+1}\cos\left[(r+1)\cos^{-1}\left(1-\frac{x}{2}\right)\right]dx + o(R^{-1}) \tag{6.15}$$

$$I_{v_1+1,v_2}(R+r,R)$$
$$= \frac{(-4)^{v_1+v_2+3}}{2R\pi}\int_0^1 \left(\frac{4-x}{x}\right)^{-1/2} x^{v_1+v_2+1}\cos\left[r\cos^{-1}\left(1-\frac{x}{2}\right)\right]dx + o(R^{-1}). \tag{6.16}$$

And third, replacing in (3.11)–(3.13) the expansions (4.2), (4.3), (6.4) and (6,5) and formulas (4.5), (4.6), (6.6) and (6.7), the constants c'_{v_1+1,v_2}, c''_{v_1+1,v_2} and c'''_{v_1+1,v_2} are found, after simplifications, to be

$$\begin{aligned} c'_{v_1+1,v_2} &= \frac{2}{(-4)^{v_1+v_2+2}} \\ c''_{v_1+1,v_2} &= -\frac{2}{(-4)^{v_1+v_2+2}} \\ c'''_{v_1+1,v_2} &= \frac{2}{(-4)^{v_1+v_2+3}}. \end{aligned}$$

Substituting these and (6.14)–(6.16) into (6.8), (6.10) and (6.12) we obtain the statements (6.1)–(6.3) of the Proposition. $\square$

7. The asymptotics of the correlation $\omega(r,u)$

Substituting the asymptotics of the double sums M_1, M_a, M_c and M_{ac} given by Propositions 5.3 and 6.1 into the formula (3.3), we obtain the following result.

PROPOSITION 7.1. *For fixed r, v_1 and v_2, we have*

$$\omega_b(R+r,v_1;R,v_2) = \frac{1}{4\pi^2}(S_1S_{ac}+S_aS_c) + o(R^{-1}),$$

where

$$S_1 = \int_0^1 \left(\frac{4-x}{x}\right)^{1/2} x^{u-1} \cos\left[r\cos^{-1}\left(1-\frac{x}{2}\right)\right] dx$$

$$S_a = \int_0^1 x^{u-1} \cos\left[(r-1)\cos^{-1}\left(1-\frac{x}{2}\right) - \frac{\pi}{2}\right] dx$$

$$S_c = \int_0^1 x^{u-1} \cos\left[(r+1)\cos^{-1}\left(1-\frac{x}{2}\right) - \frac{\pi}{2}\right] dx$$

$$S_{ac} = \int_0^1 \left(\frac{4-x}{x}\right)^{-1/2} x^{u-1} \cos\left[r\cos^{-1}\left(1-\frac{x}{2}\right)\right] dx$$

and $u = v_1 + v_2 + 2$.

REMARK 7.2. By the above result, for fixed r, v_1 and v_2 the asymptotics of $\omega_b(R + r, v_1; R, v_2)$ as $R \to \infty$ depends only on the sum $v_1 + v_2$, and not individually on v_1 and v_2. This is consistent with the expectation that the quadromer correlation at the center should depend only on the separation vector (r, u).

In the statement of Theorem 1.1, the coordinates of the separation vector (r, u) are related by $u = qr + c$, where $q \geq 0$ and c are fixed rational numbers. When $q > 0$, the asymptotics of $\omega_b(R+r, v_1; R, v_2)$ as $R \to \infty$ can be obtained from Lemma 5.1. To handle the case $q = 0$ we need the following result.

LEMMA 7.2. *Let* α*,* k *and* h *be real-valued functions that are analytic in an open interval containing* $(0, 1]$*. Assume* $\alpha'(t) > 0$ *in* $(0, 1)$ *and* $\lim_{t\to 0^+} h(t)/\alpha'(t) = 0$*. Then*

$$\int_0^1 h(t)\cos[R\alpha(t) + k(t)]dt$$
$$= \frac{h(1)}{R\alpha'(1)} \cos\left[R\alpha(1) + k(1) - \frac{\pi}{2}\right] + O(R^{-2}). \tag{7.1}$$

Proof. As in the proof of Lemma 5.1, express the integrand in terms of exponentials as

$$\int_0^1 h(t)\cos[R\alpha(t) + k(t)]dt$$
$$= \frac{1}{2}\left\{\int_0^1 e^{iR\alpha(t)} e^{ik(t)} h(t)dt + \int_0^1 e^{-iR\alpha(t)} e^{-ik(t)} h(t)dt\right\}. \tag{7.2}$$

Consider the first integral on the right hand side of (7.2). Make the change of variables $y = 1 - t$ to obtain

$$\int_0^1 e^{iR\alpha(t)} e^{ik(t)} h(t)dt = \int_0^1 e^{-iR\gamma(t)} \delta(t)dt, \tag{7.3}$$

where $\gamma(y) = -\alpha(1-y)$ and $\delta(y) = e^{ik(1-y)} h(1-y)$. These functions $\gamma(y)$ and $\delta(y)$ are readily checked to satisfy the conditions in the hypothesis of Theorem 13.2 of [**O**, p. 102] with i replaced by $-i$ throughout (clearly, by complex conjugation, the statement of the quoted theorem remains true when i is replaced by $-i$ throughout; we need to apply this modified version of the quoted theorem because its hypothesis requires $\gamma(y)' > 0$; compare with the beginning of §13.1 of [**O**]). Since the functions $\alpha(t)$, $k(t)$ and $h(t)$ of (7.1) are analytic at $t = 1$, it follows that $\gamma(y)$ and $\delta(y)$ are analytic at $y = 0$. Therefore, the exponents λ and μ of (13.02) [**O**] are both equal to 1. Thus, Theorem 13.2 of [**O**] is applicable and, since we are assuming $\lim_{t\to 0^+} h(t)/\alpha'(t) = 0$, it yields

$$\int_0^1 e^{-iR\gamma(t)}\delta(t)dt = \frac{\delta(0)}{\gamma'(0)}\frac{e^{-iR\gamma(0)}}{iR} + o(R^{-1}) \tag{7.4}$$

(since i is now replaced by $-i$ throughout Theorem 13.2 of [**O**]). By (7.3), (7.4) and the definition of $\gamma(t)$ and $\delta(t)$ we obtain that

$$\int_0^1 e^{iR\alpha(t)}e^{ik(t)}h(t)dt = \frac{e^{ik(1)}h(1)}{\alpha'(1)}\frac{e^{iR\alpha(1)}}{iR} + o(R^{-1}) \tag{7.5}$$

The two integrals on the right hand side of (7.2) are complex conjugates, so the coefficients of their asymptotic expansions are also complex conjugates. We obtain from (7.5) that

$$\int_0^1 e^{-iR\alpha(t)}e^{-ik(t)}h(t)dt = -\frac{e^{-ik(1)}h(1)}{\alpha'(1)}\frac{e^{-iR\alpha(1)}}{iR} + o(R^{-1}) \tag{7.6}$$

By (7.2)–(7.6) we obtain the statement of the Lemma. $\square$

We are now ready to prove our main result.

Proof of Theorem 1.1. Let $u = qr + c$, where $q \geq 0$ and $q, c \in \mathbb{Q}$ are fixed. Then the integral S_1 of Proposition 7.1 becomes

$$S_1 = \int_0^1 \left(\frac{4-t}{t}\right)^{1/2} t^{qr+c-1}\cos\left[r\cos^{-1}\left(1-\frac{t}{2}\right)\right]dt.$$

For $q > 0$, we can apply Lemma 5.1 with $h(t) = t^{c-1/2}(4-t)^{1/2}$, $\alpha(t) = \cos^{-1}(1-t/2)$ and $k(t) = 0$ to obtain

$$S_1 = \frac{1}{r}\frac{\sqrt{3}}{\sqrt{q^2+1/3}}\cos\left(\frac{r\pi}{3} - \arctan\frac{1}{q\sqrt{3}}\right) + o(r^{-1}), \qquad q > 0. \tag{7.7}$$

If $q = 0$, we have $u = c$ and S_1 becomes

$$S_1 = \int_0^1 \left(\frac{4-t}{t}\right)^{1/2} t^{c-1}\cos\left[r\cos^{-1}\left(1-\frac{t}{2}\right)\right]dt.$$

This has the form of the integral in (7.1), with $h(t) = t^{c-1/2}(4-t)^{1/2}$, $\alpha(t) = \cos^{-1}(1-t/2)$ and $k(t) = 0$. Clearly, $\alpha'(t) = (4t-t^2)^{-1/2} > 0$ in $(0,1)$, and it is readily checked that

$\lim_{t\to 0^+} h(t)/\alpha'(t) = 0$ (in checking this we need to use the fact that $c = u \geq 1$). Therefore, Lemma 7.2 yields

$$S_1 = \frac{3}{r}\cos\left(\frac{r\pi}{3} - \frac{\pi}{2}\right) + o(r^{-1}), \qquad q = 0. \tag{7.8}$$

A conceptual way of viewing (7.7) and (7.8) together is to say that (7.7) also holds in the limit $q \to 0^+$. Similar applications of Lemma 5.1 to the integrals S_a, S_c and S_{ac} of Proposition 7.1 yield, for $q > 0$, that

$$S_a = \frac{1}{r}\frac{1}{\sqrt{q^2+1/3}}\cos\left(\frac{r\pi}{3} - \frac{\pi}{2} - \arctan\frac{1}{q\sqrt{3}}\right) + o(r^{-1}) \tag{7.9}$$

$$S_c = \frac{1}{r}\frac{1}{\sqrt{q^2+1/3}}\cos\left(\frac{r\pi}{3} - \frac{\pi}{2} - \arctan\frac{1}{q\sqrt{3}}\right) + o(r^{-1}) \tag{7.10}$$

$$S_{ac} = \frac{1}{r}\frac{1}{\sqrt{3}\sqrt{q^2+1/3}}\cos\left(\frac{r\pi}{3} - \arctan\frac{1}{q\sqrt{3}}\right) + o(r^{-1}), \tag{7.11}$$

the formulas also holding, by Lemma 7.2, in the limit $q \to 0^+$.

By (1.4), Proposition 7.1, (7.7), (7.9)–(7.11) and the fact that the latter four relations hold also in the limit $q \to 0^+$, we obtain

$$\begin{aligned}
\omega(u,r) &= \frac{1}{4\pi^2}(S_1 S_{ac} + S_a S_c) \\
&= \frac{1}{4\pi^2 r^2(q^2+1/3)}\left\{\cos^2\!\left(\frac{r\pi}{3} - \arctan\frac{1}{q\sqrt{3}}\right) + \sin^2\!\left(\frac{r\pi}{3} - \arctan\frac{1}{q\sqrt{3}}\right)\right\} + o(r^{-2}) \\
&= \frac{3}{4\pi^2(3q^2r^2+r^2)} + o(r^{-2}) \\
&= \frac{3}{4\pi^2(r^2+3u^2)} + o(r^{-2}).
\end{aligned}$$

This proves Theorem 1.1. □

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