Supplemental problems: §2.2, §2.3

Solutions

1. Put an augmented matrix into reduced row echelon form to solve the system

\[
\begin{align*}
    x_1 - 2x_2 - 9x_3 + x_4 &= 3 \\
    4x_2 + 8x_3 - 24x_4 &= 4.
\end{align*}
\]

Solution.

\[
\begin{pmatrix}
1 & -2 & -9 & 1 & 3 \\
0 & 4 & 8 & -24 & 4
\end{pmatrix} \xrightarrow{R_2 = \frac{R_2}{4}} \begin{pmatrix}
1 & -2 & -9 & 1 & 3 \\
0 & 1 & 2 & -6 & 1
\end{pmatrix} \xrightarrow{R_1 = \frac{R_1 + 2R_2}{5}} \begin{pmatrix}
1 & 0 & -5 & -11 & 5 \\
0 & 1 & 2 & -6 & 1
\end{pmatrix}
\]

The third and fourth columns are not pivot columns, so \(x_3\) and \(x_4\) are free variables.

Our equations are

\[
\begin{align*}
    x_1 - 5x_3 - 11x_4 &= 5 \\
    x_2 + 2x_3 - 6x_4 &= 1.
\end{align*}
\]

Therefore,

\[
\begin{align*}
    x_1 &= 5 + 5x_3 + 11x_4 \\
    x_2 &= 1 - 2x_3 + 6x_4 \\
    x_3 &= x_3 \quad \text{(any real number)} \\
    x_4 &= x_4 \quad \text{(any real number)}
\end{align*}
\]

2. We can use linear algebra to find a polynomial that fits given data, in the same way that we found a circle through three specified points in the §2.1 Webwork.

Is there a degree-three polynomial \(P(x)\) whose graph passes through the points \((-2, 6), (-1, 4), (1, 6),\) and \((2, 22)\)? If so, how many degree-three polynomials have a graph through those four points? We answer this question in steps below.

a) If \(P(x) = a_0 + a_1x + a_2x^2 + a_3x^3\) is a degree-three polynomial passing through the four points listed above, then \(P(-2) = 6,\) \(P(-1) = 4,\) \(P(1) = 6,\) and \(P(2) = 22.\) Write a system of four equations which we would solve to find \(a_0,\)
\(a_1,\) \(a_2,\) and \(a_3.\)

b) Write the augmented matrix to represent this system and put it into reduced row-echelon form. Is the system consistent? How many solutions does it have?

Solution.

a) We compute

\[
\begin{align*}
    P(-2) &= 6 \quad \implies \quad a_0 + a_1 \cdot (-2) + a_2 \cdot (-2)^2 + a_3 \cdot (-2)^3 = 6, \\
    P(-1) &= 4 \quad \implies \quad a_0 + a_1 \cdot (-1) + a_2 \cdot (-1)^2 + a_3 \cdot (-1)^3 = 4, \\
    P(1) &= 6 \quad \implies \quad a_0 + a_1 \cdot 1 + a_2 \cdot 1^2 + a_3 \cdot 1^3 = 6, \\
    P(2) &= 22 \quad \implies \quad a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 + a_3 \cdot 2^3 = 22.
\end{align*}
\]
Simplifying gives us
\[
\begin{align*}
a_0 - 2a_1 + 4a_2 - 8a_3 &= 6 \\
a_0 - a_1 + a_2 - a_3 &= 4 \\
a_0 + a_1 + a_2 + a_3 &= 6 \\
a_0 + 2a_1 + 4a_2 + 8a_3 &= 22.
\end{align*}
\]

b) The corresponding augmented matrix is
\[
\begin{pmatrix}
1 & -2 & 4 & -8 & 6 \\
1 & -1 & 1 & -1 & 4 \\
1 & 1 & 1 & 1 & 6 \\
1 & 2 & 4 & 8 & 22
\end{pmatrix}
\]

We label pivots with boxes as we proceed along. First, we subtract row 1 from each of rows 2, 3, and 4.

\[
\begin{pmatrix}
1 & -2 & 4 & -8 & 6 \\
1 & -1 & 1 & 1 & 4 \\
1 & 1 & 1 & 1 & 6 \\
1 & 2 & 4 & 8 & 22
\end{pmatrix} \quad \xrightarrow{\text{row 1 subtracted}} \quad \begin{pmatrix}
1 & -2 & 4 & -8 & 6 \\
0 & 1 & -3 & 7 & -2 \\
0 & 1 & 0 & -12 & 6 \\
0 & 1 & 12 & -12 & 24
\end{pmatrix}
\]

We now create zeros below the second pivot by subtracting multiples of the second row, then divide by 6 to get

\[
\begin{pmatrix}
1 & -2 & 4 & -8 & 6 \\
0 & 1 & -3 & 7 & -2 \\
0 & 0 & 6 & -12 & 6 \\
0 & 0 & 12 & -12 & 24
\end{pmatrix} \quad \xrightarrow{\text{R}_3 = \text{R}_3 \div 6} \quad \begin{pmatrix}
1 & -2 & 4 & -8 & 6 \\
0 & 1 & -3 & 7 & -2 \\
0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}.
\]

Now we subtract a 12 times row 3 from row 4 and divide by 12:

\[
\begin{pmatrix}
1 & -2 & 4 & -8 & 6 \\
0 & 1 & -3 & 7 & -2 \\
0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 12 & 12
\end{pmatrix} \quad \xrightarrow{\text{R}_4 = \text{R}_4 \div 12} \quad \begin{pmatrix}
1 & -2 & 4 & -8 & 6 \\
0 & 1 & -3 & 7 & -2 \\
0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}.
\]

At this point we can actually use back-substitution to solve: the last row says \(a_3 = 1\), then plugging in \(a_3 = 1\) in the third row gives us \(a_2 = 3\), etc. However, for the sake of practice with reduced echelon form, let's keep row-reducing.
From right to left, we create zeros above the pivots in pivot columns by subtracting multiples of the pivot columns.

$$\begin{pmatrix}
1 & -2 & 4 & -8 & 6 \\
0 & 1 & -3 & 7 & -2 \\
0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix} \quad \begin{pmatrix}
1 & -2 & 4 & 0 & 14 \\
0 & 1 & -3 & 0 & -9 \\
0 & 0 & 1 & 0 & 3 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix} \quad \begin{pmatrix}
1 & -2 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 3 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix} \quad \begin{pmatrix}
1 & 0 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 3 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}$$

So $a_0 = 2$, $a_1 = 0$, $a_2 = 3$, and $a_3 = 1$. In other words,

$$P(x) = 2 + 3x^2 + x^3.$$ 

Therefore, there is exactly one third-degree polynomial satisfying the conditions of the problem. (You should check that, in fact, we have $P(-2) = 6$, $P(-1) = 4$, etc.)