1. Consider the augmented matrix
\[
\begin{pmatrix}
2 & -2 & 2 & | & 0 \\
1 & -3 & -4 & | & -9 \\
3 & -1 & 8 & | & 9
\end{pmatrix}
\]

**Question:** Does the corresponding linear system have a solution? If so, what is the solution set?

a) Formulate this question as a vector equation.

b) Formulate this question as a system of linear equations.

c) What does this mean in terms of spans?

d) Answer the question using the interactive demo.

e) Answer the question using row reduction.

f) Find a different solution in parts (e) and (d).

**Solution.**

a) What are the solutions to the following vector equation?
\[
x \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + y \begin{pmatrix} -2 \\ -4 \\ -1 \end{pmatrix} + z \begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ -9 \\ 9 \end{pmatrix}
\]

b) What is the solution set of the following linear system?
\[
\begin{align*}
2x - 2y + 2z &= 0 \\
x - 3y - 4z &= -9 \\
3x - y + 8z &= 9
\end{align*}
\]

c) There exists a solution if and only if \(\begin{pmatrix} 0 \\ -9 \\ 9 \end{pmatrix}\) is in \(\text{Span}\left\{\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix}\right\}\).

e) Row reducing yields
\[
\begin{pmatrix}
1 & 0 & 7/2 & | & 9/2 \\
0 & 1 & 5/2 & | & 9/2 \\
0 & 0 & 0 & | & 0
\end{pmatrix}
\]

Hence \(z\) is a free variable, so the solution in parametric form is
\[
x = \frac{9}{2} - \frac{7}{2}z \\
y = \frac{9}{2} - \frac{5}{2}z.
\]

Taking \(z = 0\) yields the solution \(x = y = 9/2\).

f) Taking \(z = 1\) yields the solution \(x = 1, y = 2\).
2. Let \( v_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \), \( v_2 = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} \), and \( w = \begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix} \).

**Question:** Is \( w \) a linear combination of \( v_1 \) and \( v_2 \)? In other words, is \( w \) in \( \text{Span}\{v_1, v_2\} \)?

a) Formulate this question as a vector equation.

b) Formulate this question as a system of linear equations.

c) Formulate this question as an augmented matrix.

d) Answer the question using the interactive demo.

e) Answer the question using row reduction.

**Solution.**

a) Does the following vector equation have a solution?

\[
x \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + y \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix}
\]

b) Does the following linear system have a solution?

\[
\begin{align*}
2x - 2y &= 2 \\
x - 3y &= -4 \\
3x - y &= 8
\end{align*}
\]

c) As an augmented matrix:

\[
\begin{pmatrix}
2 & -2 & | & 2 \\
1 & -3 & | & -4 \\
3 & -1 & | & 8
\end{pmatrix}
\]

e) Row reducing yields

\[
\begin{pmatrix}
1 & 0 & | & 7/2 \\
0 & 1 & | & 5/2 \\
0 & 0 & | & 0
\end{pmatrix}
\]

so \( x = 7/2 \) and \( y = 5/2 \).
3. Let

\[ A = \begin{pmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} \]

Is \( b \) in the span of the columns of \( A \)? In other words, is \( b \) a linear combination of the columns of \( A \)? Justify your answer.

**Solution.**

Let \( v_1, v_2, \) and \( v_3 \) be the columns of \( A \). We are asked to determine whether there are scalars \( x_1, x_2, \) and \( x_3 \) so that \( x_1v_1 + x_2v_2 + x_3v_3 = b \), which means

\[
\begin{align*}
x_1 + 5x_3 &= 2 \\
-2x_1 + x_2 - 6x_3 &= -1 \\
2x_2 + 8x_3 &= 6
\end{align*}
\]

We translate the system of linear equations into an augmented matrix, and row reduce it:

\[
\begin{pmatrix}
1 & 0 & 5 & | & 2 \\
-2 & 1 & -6 & | & -1 \\
0 & 2 & 8 & | & 6
\end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix}
1 & 0 & 5 & | & 2 \\
0 & 1 & 4 & | & 3 \\
0 & 0 & 0 & | & 0
\end{pmatrix}
\]

The right column is not a pivot column, so the system is consistent. Therefore, \( b \) is in the span of the columns of \( A \) (in other words, \( b \) is a linear combination of the columns of \( A \)).

We weren't asked to solve the equation explicitly, but if we wanted to do so, we would use the RREF of the matrix above to write

\[
x_1 = 2 - 5x_3 \quad x_2 = 3 - 4x_3 \quad x_3 = x_3 \quad (x_3 \text{ is free}).
\]

In fact, we can take \( x_1 = 2, x_2 = 3, \) and \( x_3 = 0, \) to write

\[
b = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + 0 \begin{pmatrix} 5 \\ -6 \\ 8 \end{pmatrix}.
\]

4. Consider the vector equation

\[
x \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + y \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} + z \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ -2 \end{pmatrix}.
\]

**Question:** Is there a solution? If so, what is the solution set?

**a)** Formulate this question as an augmented matrix.

**b)** Formulate this question as a system of linear equations.

**c)** What does this mean in terms of spans?

**d)** Answer the question using the interactive demo.

**e)** Answer the question using row reduction.
Solution.

a) As an augmented matrix:
\[
\begin{pmatrix}
2 & -2 & 3 & -5 \\
1 & -1 & 0 & -1 \\
3 & -1 & 4 & -2
\end{pmatrix}
\]

b) What is the solution set of the following linear system?
\[
\begin{align*}
2x - 2y + 3z &= -5 \\
x - y &= -1 \\
3x - y + 4z &= -2
\end{align*}
\]

c) There exists a solution if and only if \( \begin{pmatrix} -5 \\ -1 \\ -2 \end{pmatrix} \) is in \( \text{Span} \) \( \left\{ \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \right\} \).

e) Row reducing yields
\[
\begin{pmatrix}
1 & 0 & 0 & 3/2 \\
0 & 1 & 0 & 5/2 \\
0 & 0 & 1 & -1
\end{pmatrix},
\]
so \( x = 3/2, y = 5/2, \) and \( z = -1. \)
5. Decide if each of the following statements is true or false. If it is true, prove it; if it is false, provide a counterexample.

a) Every set of four or more vectors in \( \mathbb{R}^3 \) will span \( \mathbb{R}^3 \).

b) The span of any set contains the zero vector.

Solution.

a) This is false. For instance, the vectors

\[
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]

only span the \( x \)-axis.

b) This is true. We have

\[0 = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_p.\]

Aside: the span of the empty set is equal to \( \{0\} \), because 0 is the empty sum, i.e. the sum with no summands. Indeed, if you add the empty sum to a vector \( v \), you get \( v + (\text{no other summands}) \), which is just \( v \); and the only vector which gives you \( v \) when you add it to \( v \), is 0. (If you find this argument intriguing, you might want to consider taking abstract math courses later on.)