MATH 1553, JANKOWSKI (A1-A6)
MIDTERM 2, FALL 2018

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Write your section number here: ____________

Please **read all instructions** carefully before beginning.

- Please leave your GT ID card on your desk until your TA matches your exam.
- The maximum score on this exam is 50 points, and you have 50 minutes to complete this exam.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit! If you cannot fit your work on the front side of the page, use the back side of the page and indicate that you are using the back side.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

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Problem 1. [2 points each]

Answer true if the statement is *always* true. Otherwise, answer false. You do not need to justify your answer.

a) **T** **F** Suppose v_1, v_2, v_3 are vectors in $\mathbf{R}^4$. If $\{v_1, v_2\}$ is linearly independent and v_3 is not in $\mathrm{Span}\{v_1, v_2\}$, then $\{v_1, v_2, v_3\}$ must be linearly independent.

b) **T** **F** If A is a matrix with more rows than columns, then the matrix transformation $T(x) = Ax$ cannot be onto.

c) **T** **F** Suppose that V is a 2-dimensional subspace of $\mathbf{R}^3$ and that $\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ are in V. Then $\left\{\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\right\}$ is a basis for V.

d) **T** **F** If A is a 3×3 matrix, then Col A must contain the vector $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

e) **T** **F** Suppose $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation with standard matrix A. If T is not one-to-one, then $Ax = 0$ must have infinitely many solutions.

Solution.

a) True: The span of $\{v_1, v_2, v_3\}$ will be larger than the planned spanned by v_1 and v_2, so $\mathrm{Span}\{v_1, v_2, v_3\}$ will be 3-dimensional subspace of $\mathbf{R}^4$ and thus $\{v_1, v_2, v_3\}$ is linearly independent.

b) True: A cannot have more than one pivot in any row or column, so since it has more rows than columns, it can't have a pivot in every row.

c) True: The two vectors are linearly independent vectors in a 2-dimensional subspace, so they are a basis for V by the Basis Theorem.

d) True: Col$A$ is a subspace of $\mathbf{R}^3$, and every subspace contains the zero vector.

e) True. If T is not one-to-one then A fails to have a pivot in some column, so $Ax = 0$ will have a free variable and thus infinitely many solutions.

Scrap paper for problem 1

Problem 2.

[11 points]

Short answer. In this problem, you don't need to justify your answers.

a) Which of the following are subspaces of $\mathbf{R}^3$? Circle all that apply.

(i) The set of solutions to the equation $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 4 \\ 1 & 0 & 3 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$\boxed{\text{(ii)}}$ $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; 2x - y + z = 0 \right\}$

(iii) The set $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

b) Let A be a 4×6 matrix, and let T be the matrix transformation $T(x) = Ax$. Which of the following are **possible**? Circle all that apply.

(i) Nul A is a line through the origin.

$\boxed{\text{(ii)}}$ For every b in $\mathbf{R}^4$, the equation $Ax = b$ is consistent.

(iii) $\dim(\text{Col } A) = 6$.

(iv) For some b in $\mathbf{R}^4$, the equation $T(x) = b$ has a unique solution.

(v) For every b in $\mathbf{R}^4$, the equation $T(x) = b$ has at most one solution.

c) Fill in the blank: The dimension of Span $\left\{ \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$ is __2__

d) Write a 2×2 matrix A that is not the identity matrix and that satisfies $A^2 = I$.

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is one example. Another is $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Solution.

Here we'll explain the answers for problem 2 in depth.

a) In (i), the set is not a subspace because it doesn't contain the zero vector. This is actually extra practice problem #1(c), just slightly rephrased.
In (ii), W is a subspace. In fact, W is the null space of $\begin{pmatrix} 2 & -1 & 1 \end{pmatrix}$.
The set in (iii) is not closed under addition or scalar multiplications. For example, it does not contain $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$.

b) By the Rank Theorem, $\dim(\text{Col } A) + \dim(\text{Nul} A) = 6$, and Col A is a subspace of $\mathbf{R}^4$ so $\dim(\text{Col } A) \leq 4$.
(i) is impossible since that would mean $\dim(\text{Nul}A) = 1$ and $\dim(\text{Col}A) = 5$.
Alternate explanation: Since A is 4×6 it has at most 4 pivots, so the equation $Ax = 0$ will have at least two variables and Nul$A$ will be at least a plane.
(ii) is possible since it just means T is onto (write a 4×6 matrix with 4 pivots).
(iii) is impossible: Col A is a subspace of $\mathbf{R}^4$, so it can't be 6-dimensional.
(iv) is impossible: A has more columns than rows, so T cannot be one-to-one, thus $T(x) = 0$ has infinitely many solutions. Any consistent system $T(x) = b$ is a translate of the homogeneous solution set, and thus will have infinitely many solutions.
(v) is impossible: $T(x) = 0$ must have infinitely many solutions since A can't have a pivot in every column.

c) The dimension is 2 because V is the span of the two linearly independent vectors $\left\{\begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\right\}$.

d) Straight from the supplemental problems list and also was done in class.

One example is $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (reflection about $y = x$), which satisfies $A^2 = I$ because performing A twice sends $(x, y) \to (y, x) \to (x, y)$.

Another example is $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, the matrix rotation by 180° (or equivalently, $A = -I$), since rotating twice by 180° gets you right back where you started.

Many other examples are possible, for example $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The most creative correct student answer was something like $A = \begin{pmatrix} -1 & 0 \\ \frac{9999}{7} & 1 \end{pmatrix}$.

Space for extra work on problem 2

Problem 3.

[8 points]

Burt Macklin has row-reduced the matrix A below into its RREF.

$$A = \begin{pmatrix} 1 & -2 & 3 & 3 \\ 2 & -4 & 6 & 2 \\ -1 & 2 & -3 & 1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) Write a basis for Col A. (no justification required for this part)

We use the pivot columns of A: $\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}.$

b) Find a basis for Nul A.

We put the system $Ax = 0$ into parametric vector form. The system is

$$x_1 - 2x_2 + 3x_3 = 0, \qquad x_4 = 0,$$

so x_2 and x_3 are free, and

$$x_1 = 2x_2 - 3x_3, \qquad x_4 = 0$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ x_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -3x_3 \\ 0 \\ x_3 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

A basis for Nul A is $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$

c) Let T be the matrix transformation $T(x) = Ax$. Is there a vector in the codomain of T which is not in the range of T? Briefly justify your answer.

Yes. This question is really asking us to determine whether T is onto. Since range$(T) = \text{Col}(A)$ and Col(A) is a 2-dimensional subspace of $\mathbf{R}^3$ by part (a), we see T is not onto. In other words, there is a vector in the codomain $\mathbf{R}^3$ which is not in the range of T.

Space for extra work on problem 3

Problem 4.

[11 points]

Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be the transformation

$$T(x, y, z) = (x - 3y + z,\ z - 2x),$$

and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation of *clockwise* rotation by 45°.

a) Write the standard matrix A for T and the standard matrix B for U. Simplify your answers (do not leave anything in terms of sine or cosine).

$$A = \begin{pmatrix} T(e_1) & T(e_2) & T(e_3) \end{pmatrix} = \begin{pmatrix} 1 & -3 & 1 \\ -2 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} \cos(-45^\circ) & -\sin(-45^\circ) \\ \sin(-45^\circ) & \cos(-45^\circ) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Alternatively, we could have computed B quickly by using that

$$B = \begin{pmatrix} U(e_1) & U(e_2) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

b) Circle all transformations that are one-to-one. T $\boxed{U}$

c) Circle the composition that makes sense: $\boxed{U \circ T}$ $T \circ U$

d) Compute the standard matrix of the composition you circled in (c).

$$BA = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & 1 \\ -2 & 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -3 & 2 \\ -3 & 3 & 0 \end{pmatrix}.$$

Space for extra work on problem 4

Problem 5.

[10 points]

Parts (a) and (b) are unrelated.

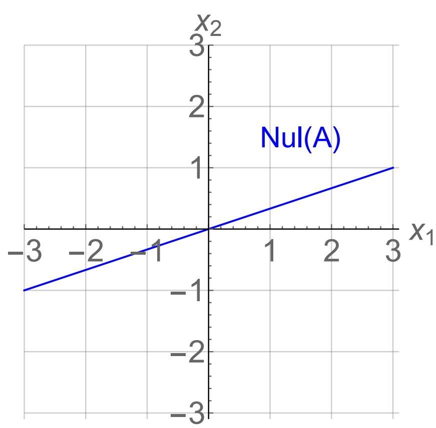
a) Find all values of h so that $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ h \\ 5 \end{pmatrix} \right\}$ is linearly independent.

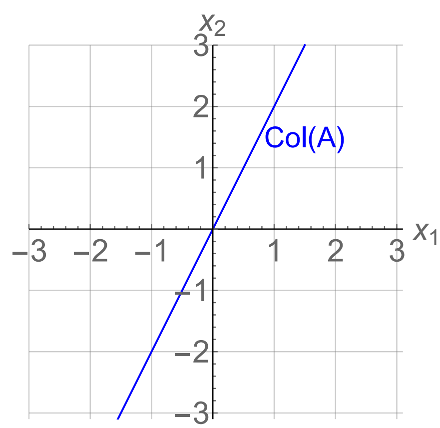
$$\begin{pmatrix} 1 & 3 & 4 \\ -1 & -2 & h \\ 0 & 4 & 5 \end{pmatrix} \xrightarrow{R_2=R_2+R_1} \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & h+4 \\ 0 & 4 & 5 \end{pmatrix} \xrightarrow{R_3=R_3-4R_2} \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & h+4 \\ 0 & 0 & 5-4(h+4) \end{pmatrix}.$$

The matrix has 3 pivots if and only if

$$5-4(h+4) \neq 0 \qquad -4h-11 \neq 0 \qquad \boxed{h \neq -\frac{11}{4}}.$$

b) Find a matrix A whose null space and column space are drawn below:





We need Col $A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ and Nul $A = \text{Span}\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$ (the line $x_1 = 3x_2$).

We could make the first column of A equal to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and then note that Nul A will be the line $x_1 = 3x_2$ as long as the second column is -3 times the first.

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}.$$

Other examples are possible, for example $A = \begin{pmatrix} -1 & 3 \\ -2 & 6 \end{pmatrix}$ and $A = \begin{pmatrix} 2 & -6 \\ 4 & -12 \end{pmatrix}$.

Space for extra work on problem 5