MATH 1553
FINAL EXAM, FALL 2018

Name	

Circle the name of your instructor below:

Bonetto   Brito 1:55-2:45 PM   Brito 3:00-3:50 PM

Duan   Jankowski   Kordek

Margalit 11:15 AM -12:05 PM   Margalit 12:20-1:10 PM   Rabinoff

Srinivasan 3:00-3:50 PM   Srinivasan 4:30-5:20 PM

DO NOT WRITE IN THE TABLE BELOW. It will be used to record scores.

1	2	3	4	5	6	7	8	9	10	Total

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 100 points.
- You have 170 minutes to complete this exam.
- You may not use any calculators or aids of any kind (notes, text, etc.).
- Unless a problem specifies that no work is required, show your work or you may receive little or no credit, even if your answer is correct.
- If you run out of room on a page, you may use its back side to finish the problem, but please indicate this.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Check your answers if you have time left! Most linear algebra computations can be easily verified for correctness. Good luck!

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam.

____

Problem 1.

[1 point each]

True or false. Circle **T** if the statement is *always* true. Otherwise, answer **F**. You do not need to justify your answer. In every case, A is a matrix whose entries are real numbers.

a) **T** **F** Suppose $\{v_1, \ldots, v_6\}$ is a set of vectors that spans $\mathbb{R}^5$. Then $\{v_1, \ldots, v_6\}$ is a basis for $\mathbb{R}^5$.

b) **T** **F** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a one-to-one linear transformation. Then $n \leq m$.

c) **T** **F** Suppose A is an $n \times n$ matrix and the sum of the columns of A is the zero vector. Then A is not invertible.

d) **T** **F** Suppose A is a square matrix that is diagonalizable and invertible. Then A^{-1} is diagonalizable.

e) **T** **F** Suppose A is a 3×3 matrix with characteristic polynomial $-\lambda^3 - \lambda^2 - \lambda - 1$. Then A is invertible.

f) **T** **F** Suppose A is a 3×3 matrix whose characteristic polynomial is $-\lambda^3 - \lambda^2$ and whose null space is a line. Then A is diagonalizable.

g) **T** **F** There is a 2×2 matrix A so that the solution set of $Ax = \binom{1}{2}$ is the line $y = 2x + 1$ and the solution set of $Ax = \binom{-4}{1}$ is the line $y = 3x - 1$.

h) **T** **F** If A is the standard matrix for an orthogonal projection onto a subspace, then

$$(\mathrm{Nul}\, A)^{\perp} = \mathrm{Col}\, A.$$

i) **T** **F** Let T be the linear transformation given by orthogonal projection onto the subspace

$$W = \left\{(x, y, z, w) \text{ in } \mathbf{R}^4 \mid x + y + z + 2w = 0\right\}.$$

Then the dimension of the range of T is 3.

j) **T** **F** There is a stochastic 2×2 matrix A that has $1 + i$ as an eigenvalue.

Solution to problem 1, "Exam" version.

a) False. Every basis of $\mathbf{R}^5$ consists of exactly 5 vectors.

b) True. A matrix cannot have more pivots than it has rows or columns. If $m < n$, then the $m \times n$ standard matrix A for T can only have a max of m pivots, which means it must have fewer than n pivots and thus cannot have a pivot in every column.

c) True. The columns of A are linearly dependent since $v_1 + \cdots + v_n = 0$, so by the Invertible Matrix Theorem A is not invertible.

d) True. If $A = CDC^{-1}$ for an invertible C and diagonal D then

$$A^{-1} = (CDC^{-1})^{-1} = CD^{-1}C^{-1}.$$

e) True: $\det(A) = \det(A - 0I) = 0^3 - 0^2 - 0 - 1 = -1$, so $\det(A) \neq 0$ hence A is invertible.

f) False: $-\lambda^3 - \lambda^2 = -\lambda^2(\lambda + 1)$, so $\lambda = 0$ has algebraic multiplicity 2 but geometric multiplicity 1, therefore A is not diagonalizable.

g) False. Solution sets to $Ax = b$ must be translations of (and thus parallel to) the solution set to $Ax = 0$, but the lines $y = 2x + 1$ and $y = 3x - 1$ are not parallel.

h) True. Let W be the subspace. Then $\mathrm{Nul} A = W^\perp$ and $\mathrm{Col}\ A = W$, so $(\mathrm{Nul} A)^\perp = W = \mathrm{Col}\ A$.

i) True. $W = \mathrm{Nul}\begin{pmatrix} 1 & 1 & 1 & 2 \end{pmatrix}$ hence $\dim(W) = 3$. Since $W = \mathrm{range}(T)$ this means $\dim(\mathrm{range}\, T) = 3$.

j) False. Since $\lambda = 1$ is automatically an eigenvalue of any stochastic matrix we know 1 is an eigenvalue of A. If $1 + i$ were an eigenvalue of A then so would $1 - i$, so the 2×2 matrix would have three distinct eigenvalues.

Problem 2. [2 points each]

Short answer. You do not need to show your work, and there is no partial credit. In each case, A is a matrix whose entries are real numbers.

a) Suppose that A is a 2×2 matrix, that 5 is an eigenvalue of A, and that A is not diagonalizable. What is the characteristic polynomial of A?

b) Find a 2×2 matrix whose column space is the line $y = 2x$ and whose null space is the x-axis.

c) Suppose u and v are orthogonal vectors with $||u|| = 2$ and $||v|| = 3$. Compute the dot product $(4u + 5v) \cdot v$.

d) Let A be a 3×3 matrix whose 3-eigenspace is a two-dimensional plane and whose 1-eigenspace is a line. What is the determinant of A?

e) Suppose that $\det\begin{pmatrix} a & b & c \\ 1 & 2 & 3 \\ 0 & -1 & 5 \end{pmatrix} = 2$. Find $\det\begin{pmatrix} 1 & 2 & 3 \\ 5a+1 & 5b+2 & 5c+3 \\ 1 & 1 & 8 \end{pmatrix}$.

Solution to problem 2, “Exam" version.

a) A can only have the one eigenvalue $\lambda = 5$, otherwise it would have two distinct eigenvalues and consequently be diagonalizable. Therefore, $\lambda = 5$ must have algebraic multiplicity 2, so

$$\det(A - \lambda I) = (5-\lambda)^2 = (\lambda - 5)^2 = \lambda^2 - 10\lambda + 25.$$

All of the above answers are acceptable.

b) The first column must be zero and the second column must be any nonzero multiple of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. For example,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & -1 \\ 0 & -2 \end{pmatrix}, \qquad \text{etc.}$$

c) $(4u + 5v) \cdot v \;=\; (4u \cdot v) + (5v \cdot v) \;=\; 4(0) + 5(9) = 45.$

d) Two ways of seeing this: one way is to note that

$$\det(A - \lambda I) = (3-\lambda)^2(1-\lambda) \qquad \text{so} \qquad \det(A) = 3^2(1) = 9.$$

Another way: Since A is diagonalizable we have $A = CDC^{-1}$ where $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, so

$$\det(A) = \det(CDC^{-1}) = \det(C)\det(D)\det(C^{-1}) = \det(C) \cdot 9 \cdot \frac{1}{\det(C)} = 9.$$

e) Aside of the two row replacements (which don't affect determinants), the new matrix is formed by doing one row swap and multiplying a row by 5, so its determinant is $2(-1)(5) = -10$.

Problem 3.

[(a), (b), (c), (d) are worth 1, 2, 3, 4 points respectively]

Short answer. You do not need to show your work, and there is no partial credit.

a) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be a linear transformation and suppose $\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ are in the range of T. Write another nonzero vector in the range of T here: $\begin{pmatrix} \quad \end{pmatrix}$

b) Suppose that A is a 12×9 matrix and the solution set to $Ax = 0$ has dimension 7.

(i) Fill in the blank: the dimension of the column space of A is ________.

(ii) Fill in the blank: the dimension of the row space of A is ________.

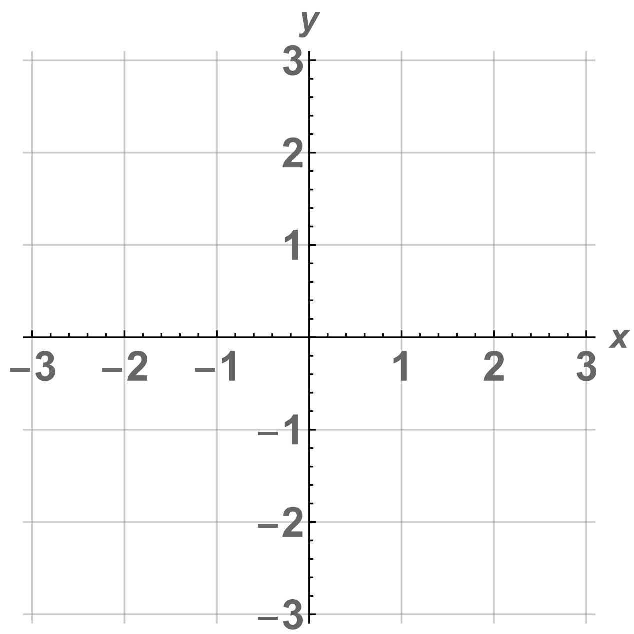
c) Suppose A is a stochastic matrix. Which of the following must be true? Circle all that apply.

(i) The sum of entries in each row of A is equal to 1.

(ii) The sum of entries in each column of A is equal to 1.

(iii) No entry of A is greater than 1.

d) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation of reflection across the line $y = 3x$, and let A be the standard matrix for T. Draw each eigenspace of A precisely, and clearly label each eigenspace with its eigenvalue.



Solution to problem 3, “Exam" version.

a) Any vector in $\mathbf{R}^3$ that has 0 as its third entry is in the range of T. For example, $\begin{pmatrix} 10 \\ 1 \\ 0 \end{pmatrix}$.

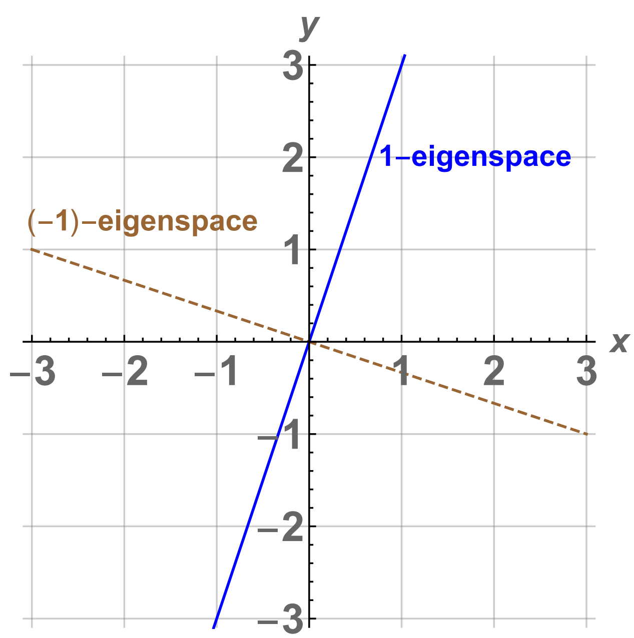
b) By the Rank Theorem,

$$\dim(\text{Col } A) + \dim(\text{Nul } A) = 9.$$

We've been given $\dim(\text{Nul } A) = 7$, so
(i) $\dim(\text{Col } A) = 2$
(ii) $\dim(\text{Row } A) = \dim((\text{Nul} A)^{\perp}) = 9 - 7 = 2$.

c) (ii) must be true: it is part of the definition of stochastic matrix.
(iii) is also true since each column must sum to 1 and all entries must be 0 or greater (thus no entry can be larger than 1).

d) The reflection fixes the line $y = 3x$, so the 1-eigenspace is the line $y = 3x$. The reflection geometrically flips the line perpendicular to $y = 3x$, so the other eigenvalue is $\lambda = -1$, and the (-1)-eigenspace is the (perpendicular) line $y = -x/3$.



Problem 4.

[2 points for (a); 4 points each for (b) and (c)]

No work is necessary in parts (a) and (b). Show your work in part (c).

a) Complete the following definition.
A vector v in $\mathbf{R}^n$ is an *eigenvector* of an $n \times n$ matrix A if . . .

b) Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with standard matrix A. Which of the following conditions guarantee that T *must* be one-to-one? Circle all that apply.

(i) A has m pivots.

(ii) The columns of A are linearly independent.

(iii) For each input vector x in $\mathbf{R}^n$, there is exactly one output vector $T(x)$ in $\mathbf{R}^m$.

(iv) The equation $Ax = b$ has exactly one solution for each b in $\mathbf{R}^m$.

c) The inverse of $A = \begin{pmatrix} 1 & 3 & 5 \\ -1 & -4 & -8 \\ 1 & 5 & 12 \end{pmatrix}$ is $A^{-1} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$.

Solution to problem 4.

a) v is not the zero vector and $Av = \lambda v$ for some scalar λ.

b) The correct answers are (ii) and (iv).
(ii) is equivalent to T being one-to-one, and (iv) guarantees T is one-to-one and onto. However, (i) is not necessarily true and (iii) is just the definition of a function.

c) We row-reduce $\begin{pmatrix} A & I \end{pmatrix}$:

$$\begin{pmatrix} 1 & 3 & 5 & | & 1 & 0 & 0 \\ -1 & -4 & -8 & | & 0 & 1 & 0 \\ 1 & 5 & 12 & | & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_3=R_3-R_1]{R_2=R_2+R_1} \begin{pmatrix} 1 & 3 & 5 & | & 1 & 0 & 0 \\ 0 & -1 & -3 & | & 1 & 1 & 0 \\ 0 & 2 & 7 & | & -1 & 0 & 1 \end{pmatrix} \xrightarrow[\text{then } R_2=-R_2]{R_3=R_3+2R_2} \begin{pmatrix} 1 & 3 & 5 & | & 1 & 0 & 0 \\ 0 & 1 & 3 & | & -1 & -1 & 0 \\ 0 & 0 & 1 & | & 1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[R_1=R_1-5R_3]{R_2=R_2-3R_3} \begin{pmatrix} 1 & 3 & 0 & | & -4 & -10 & -5 \\ 0 & 1 & 0 & | & -4 & -7 & -3 \\ 0 & 0 & 1 & | & 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_1=R_1-3R_2} \begin{pmatrix} 1 & 0 & 0 & | & 8 & 11 & 4 \\ 0 & 1 & 0 & | & -4 & -7 & -3 \\ 0 & 0 & 1 & | & 1 & 2 & 1 \end{pmatrix}.$$

We have found $A^{-1} = \begin{pmatrix} 8 & 11 & 4 \\ -4 & -7 & -3 \\ 1 & 2 & 1 \end{pmatrix}$.

Free response. For all problems remaining, show all work, and justify your answers where appropriate. A correct answer without proper work may receive little or no credit.

Problem 5.

Parts (a) and (b) are unrelated.

a) [6 points] Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation obtained by reflecting over the line $y = 0$ and then rotating by $45°$ counterclockwise.

(i) Find the standard matrix for T. Write your answer here: $\begin{pmatrix} & \\ & \end{pmatrix}$

(ii) Is T one-to-one? Justify your answer.

b) [4 points] Define linear transformations $S : \mathbf{R}^3 \to \mathbf{R}^2$ and $U : \mathbf{R}^2 \to \mathbf{R}^3$ by

$$S(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, x_3) \quad \text{and} \quad U(x_1, x_2) = (x_1 + x_2, 3x_1 - x_2, x_1).$$

Find the standard matrix for $S \circ U$. Write your answer here: $\begin{pmatrix} & \\ & \end{pmatrix}$

Solution to problem 5.

a) (i) We could compute the matrix as

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Alternatively, we could look at each basis vector's image.

$$e_1 \to e_1 \to \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad e_2 \to -e_2 \to \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Thus our matrix is

$$\begin{pmatrix} T(e_1) & T(e_2) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

(ii) Yes: T is a composition of invertible functions and is thus invertible, so it is one-to-one. Alternatively, T is one-to-one because its matrix has a pivot in every column.

b)

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$

Problem 6.

Your roommate Karxon has given you the following matrix A and its reduced row echelon form:

$$A = \begin{pmatrix} 1 & 2 & -1 & -1 \\ -1 & -2 & 2 & 5 \\ 2 & 4 & 0 & 6 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) [4 points] Find a basis for Nul A.

b) [6 points] Find the closest vector w to $\begin{pmatrix} -3 \\ -2 \\ 5 \end{pmatrix}$ in Col A.

$w = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$

Solution to problem 6, “Exam" version.

a) The RREF of A gives us

$$\begin{aligned} x_1 &= -2x_2 - 3x_4 \\ x_2 &= x_2 \\ x_3 &= -4x_4 \\ x_4 &= x_4. \end{aligned}$$

So

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_2 - 3x_4 \\ x_2 \\ -4x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ -4 \\ 1 \end{pmatrix}. \qquad \text{Basis for Nul } A\text{: } \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -4 \\ 1 \end{pmatrix} \right\}.$$

b) Let $B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \\ 2 & 0 \end{pmatrix}$, so the columns of B are a basis of Col$A$.

We solve $B^T B v = B^T b$ where $b = \begin{pmatrix} -3 \\ -2 \\ 5 \end{pmatrix}$.

$$B^T B = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -3 & 5 \end{pmatrix},$$

$$B^T b = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \end{pmatrix}.$$

$$\left(B^T B \quad B^T \mid b \right) = \left(\begin{array}{cc|c} 6 & -3 & 9 \\ -3 & 5 & -1 \end{array} \right) \xrightarrow{R_2 = R_2 + \frac{R_1}{2}} \left(\begin{array}{cc|c} 6 & -3 & 9 \\ 0 & 7/2 & 7/2 \end{array} \right) \xrightarrow{\text{scale } R_2} \left(\begin{array}{cc|c} 6 & -3 & 9 \\ 0 & 1 & 1 \end{array} \right)$$

$$\xrightarrow{R_1 = R_1 + 3R_2} \left(\begin{array}{cc|c} 6 & 0 & 12 \\ 0 & 1 & 1 \end{array} \right) \xrightarrow{R_1 = R_1/6} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array} \right), \qquad v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Therefore,

$$w = Bv = \begin{pmatrix} 1 & -1 \\ -1 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}.$$

Problem 7.

Let $A = \begin{pmatrix} 3 & 5 & 2 \\ 0 & 2 & 0 \\ 1 & -2 & 4 \end{pmatrix}$. Its eigenvalues are $\lambda = 2$ and $\lambda = 5$.

a) [5 points] Find a basis for each eigenspace of A. Enter your answers below.

Basis for 2-eigenspace: $\left\{ \qquad \right\}$ Basis for 5-eigenspace: $\left\{ \qquad \right\}$.

b) [3 points] Is A diagonalizable? If your answer is yes, write an invertible matrix C and diagonal matrix D so that $A = CDC^{-1}$. If your answer is no, justify why A is not diagonalizable.

c) [2 points] Find a basis for $(\mathrm{Nul} A)^{\perp}$. Write your answer here: $\left\{ \qquad \right\}$.

Solution to problem 7.

a) For $\lambda = 2$:

$$\left(A-2I \mid 0\right) = \left(\begin{array}{ccc|c} 1 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -2 & 2 & 0 \end{array}\right) \xrightarrow{R_3=R_3-R_1} \left(\begin{array}{ccc|c} 1 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \end{array}\right) \xrightarrow[\text{then } R_1=R_1-5R_3,\ R_2\leftrightarrow R_3]{R_3=-\frac{R_3}{7}} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus $x_1 = -2x_3,\ x_2 = 0$, and x_3 is free. A basis for the 2-eigenspace is $\left\{\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}\right\}$.

For $\lambda = 5$:

$$\left(A-5I \mid 0\right) = \left(\begin{array}{ccc|c} -2 & 5 & 2 & 0 \\ 0 & -3 & 0 & 0 \\ 1 & -2 & -1 & 0 \end{array}\right) \xrightarrow[R_2=-\frac{R_2}{3}]{R_1\leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 5 & 2 & 0 \end{array}\right) \xrightarrow{R_3=R_3+2R_1} \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\text{Clear steps}} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus $x_1 = x_3,\ x_2 = 0$, and x_3 is free. A basis for the 5-eigenspace is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

b) No, A only has two linearly independent eigenvectors, so A is not diagonalizable.

c) We've been told the eigenvalues of A are $\lambda = 2$ and $\lambda = 5$, so 0 is not an eigenvalue of A, thus A is invertible and Nul $A = \{0\}$.

Since $\dim(\mathrm{Nul}A) + \dim((\mathrm{Nul}A)^{\perp}) = \dim(\mathbf{R}^3) = 3$, we see $(\mathrm{Nul}A)^{\perp} = \mathbf{R}^3$. Any basis of $\mathbf{R}^3$ will be a basis for $(\mathrm{Nul}A)^{\perp}$. An example is

$$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

Problem 8.

Let $A = \begin{pmatrix} -1 & -2 \\ 5 & 5 \end{pmatrix}$.

a) [6 points] Find the characteristic polynomial of A and the eigenvalues of A. Write your answers for the eigenvalues in the spaces below.
The eigenvalue with *positive* imaginary part is $\lambda_1 =$ ____________.

The eigenvalue with *negative* imaginary part is $\lambda_2 =$ ____________.

b) [4 points] For each eigenvalue of A, find a corresponding eigenvector.
Write your answers below:

An eigenvector for λ_1 (the eigenvalue with *positive* imaginary part) is $v_1 = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$.

An eigenvector for λ_2 (the eigenvalue with *negative* imaginary part) is $v_2 = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$.

Solution to problem 8, “Exam" version.

a) The characteristic polynomial of A is

$$\det(A-\lambda I)=\lambda^2-\mathrm{Tr}(A)\lambda+\det(A)=\lambda^2-4\lambda+5.$$

Solving $\lambda^2-4\lambda+5=0$ gives

$$\lambda=\frac{4\pm\sqrt{16-4(1\cdot 5)}}{2}=\frac{4\pm\sqrt{-4}}{2}=\frac{4\pm 2i}{2}=2\pm i.$$

Thus, $\lambda_1=2+i$ and $\lambda_2=2-i$.

b) For $\lambda_1=2+i$, we find

$$\left(A-(2+i)I \mid 0\right)=\left(\begin{array}{cc|c}-3-i & -2 & 0\\ \star & \star & \star\end{array}\right).$$

The first row has the form $\begin{pmatrix}a & b\end{pmatrix}$ so the old trick gives eigenvector:

$$v_1=\begin{pmatrix}-b\\ a\end{pmatrix}=\begin{pmatrix}2\\ -3-i\end{pmatrix}\quad\text{or equivalently}\quad v_1=\begin{pmatrix}b\\ -a\end{pmatrix}=\begin{pmatrix}-2\\ 3+i\end{pmatrix}.$$

Alternatively, we could row-reduce:

$$\left(A-(2+i)I \mid 0\right)=\left(\begin{array}{cc|c}-3-i & -2 & 0\\ \star & \star & \star\end{array}\right)\xrightarrow[\text{destroy } R_2]{R_1=\frac{R_1}{-3-i}}\left(\begin{array}{cc|c}1 & \frac{-2}{-3-i} & 0\\ 0 & 0 & 0\end{array}\right)=\left(\begin{array}{cc|c}1 & \frac{3-i}{5} & 0\\ 0 & 0 & 0\end{array}\right).$$

So $x_1=\frac{-3+i}{5}x_2$ and $v_1=\begin{pmatrix}\frac{-3+i}{5}\\ 1\end{pmatrix}$.

All these answers are nonzero complex multiples of each other, so all are correct.

An eigenvector for $\lambda_2=2-i$ is $v_2=\overline{v_1}$, so

$$v_2=\overline{v_1}=\begin{pmatrix}2\\ -3+i\end{pmatrix},$$

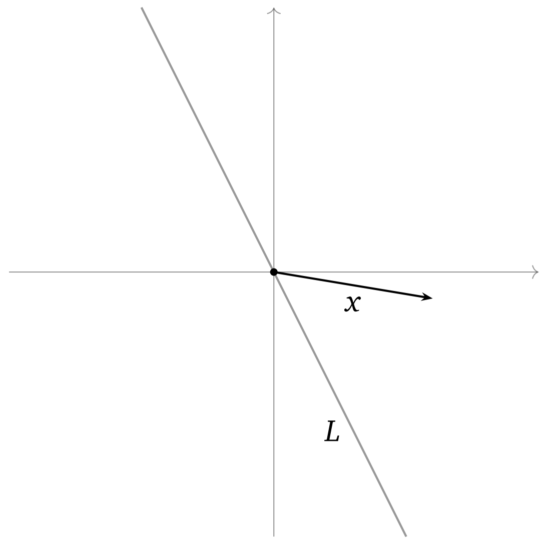
or $v_2=\begin{pmatrix}-2\\ 3-i\end{pmatrix}$, or $v_2=\begin{pmatrix}\frac{-3-i}{5}\\ 1\end{pmatrix}$.

Problem 9.

Let L be the line in $\mathbf{R}^2$ spanned by $u = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Recall our notation: if x is a vector, then x_L is the orthogonal projection of x onto L.

a) [3 points] Let x be the vector graphed below.
Carefully sketch three things: $L^\perp$, x_L, and $x_{L^\perp}$. Clearly label each.

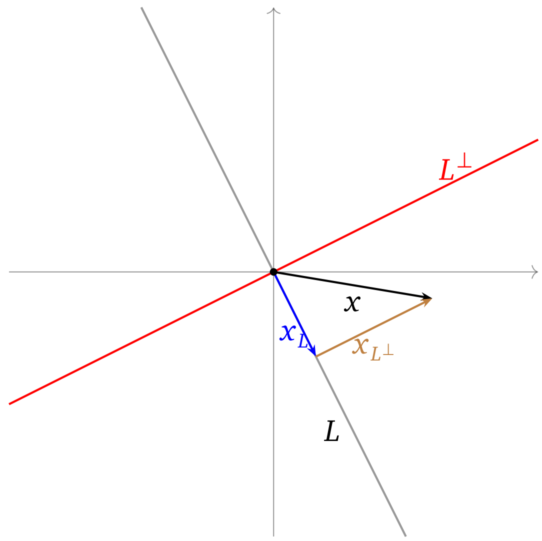


b) [4 points] Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation given by orthogonal projection onto L. Find the standard matrix for T.

c) [3 pts] Compute y_L and $y_{L^\perp}$ for $y = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$. $y_L = \begin{pmatrix} \quad \end{pmatrix}$ $y_{L^\perp} = \begin{pmatrix} \quad \end{pmatrix}$

Solution to problem 9, “Exam" version.

a) L is the line $y = \frac{x}{2}$.



b) The matrix is

$$B = \frac{1}{u \cdot u} u u^T = \frac{1}{1+4} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \begin{pmatrix} 1 & -2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}.$$

It’s fine to leave it like the above or to pull the fraction $\frac{1}{5}$ into the matrix to get

$$B = \begin{pmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{pmatrix}.$$

c)

$$y_L = By = \begin{pmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

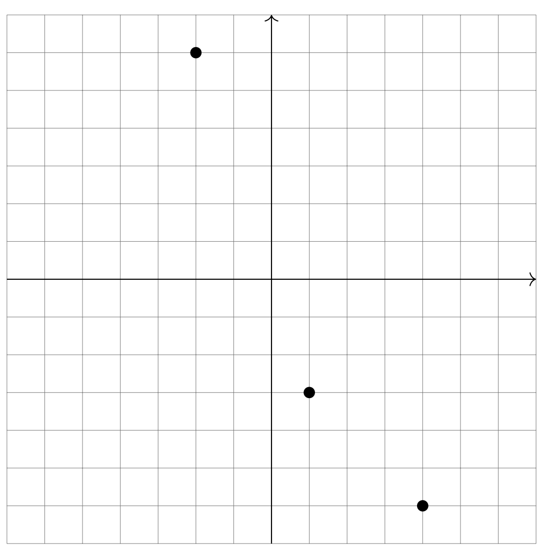
$$y_{L^\perp} = y - y_L = \begin{pmatrix} 5 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}.$$

Problem 10.

Consider the data points $(-2, 6)$, $(1, -3)$, and $(4, -6)$. Find the best-fit line for these data points. Enter your answer in the space below.

$y = \underline{\quad -2 \quad} x + \underline{\quad 1 \quad}.$

For your benefit, the data points are plotted at the bottom of the page, so that you may check your answer by plotting your line to make sure it looks reasonable (the graph will not be graded; it is there solely for you to check your work).



Solution to problem 10.

It doesn't matter the order we write the terms, as long as we keep them in the correct order in the end.

$$y = Mx + B$$

$$\begin{aligned} x = -2,\ y = 6: &\quad 6 = -2M + B \\ x = 1,\ y = -3: &\quad -3 = M + B \\ x = 4,\ y = 6: &\quad -6 = 4M + B \end{aligned}$$

For $A = \begin{pmatrix} -2 & 1 \\ 1 & 1 \\ 4 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 6 \\ -3 \\ -6 \end{pmatrix}$, we solve $A^T A\widehat{x} = A^T b$. Our order is $\widehat{x} = \begin{pmatrix} M \\ B \end{pmatrix}$.

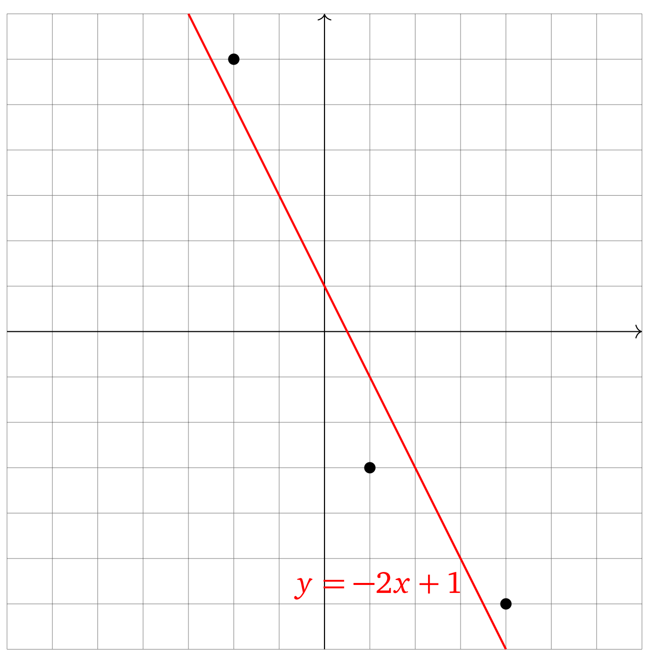
$$A^T A = \begin{pmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 21 & 3 \\ 3 & 3 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ -3 \\ -6 \end{pmatrix} = \begin{pmatrix} -39 \\ -3 \end{pmatrix}.$$

We solve $A^T A\widehat{x} = A^T b$.

$$\left(\begin{array}{cc|c} 21 & 3 & -39 \\ 3 & 3 & -3 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 3 & 3 & -3 \\ 21 & 3 & -39 \end{array}\right) \xrightarrow[R_1 = \frac{R_1}{3}]{R_2 = R_2 - 7R_1} \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & -18 & -18 \end{array}\right) \xrightarrow[\text{then } R_1 = R_1 - R_2]{R_2 = -\frac{R_2}{18}} \left(\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 1 \end{array}\right).$$

Thus $\widehat{x} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, so $y = -2x + 1$.



Scrap paper. This page will not be graded under any circumstances