Math 1553 Worksheet §2.2, §2.3

Solutions

1. Is it possible for a linear system to have a unique solution if it has more equations than variables? If yes, give an example. If no, justify why it is impossible.

Solution.

It is possible. One example is the system below, which has unique solution $x = 5$, $y = 2$:

$$\begin{aligned} x + y &= 7 \\ x - y &= 3 \\ 2x + 2y &= 14. \end{aligned}$$

2. **a)** Which of the following matrices are in row echelon form? Which are in reduced row echelon form?

b) For the matrices in row echelon form, which entries are the pivots? What are the pivot columns?

$$\begin{pmatrix} \color{red}{1} & \color{blue}{0} & \color{blue}{0} & 0 \\ \color{blue}{0} & \color{red}{1} & \color{blue}{0} & 0 \\ \color{blue}{0} & \color{blue}{0} & \color{red}{1} & 1 \end{pmatrix} \qquad \begin{pmatrix} \color{red}{1} & \color{blue}{0} & 1 & \color{blue}{0} \\ \color{blue}{0} & \color{red}{1} & 1 & \color{blue}{0} \\ \color{blue}{0} & \color{blue}{0} & 0 & \color{red}{1} \end{pmatrix} \qquad \begin{pmatrix} \color{red}{1} & \color{blue}{1} & 0 & \color{blue}{1} & \color{blue}{1} \\ \color{blue}{0} & \color{red}{2} & 0 & \color{blue}{2} & \color{blue}{2} \\ \color{blue}{0} & \color{blue}{0} & 0 & \color{red}{3} & \color{blue}{3} \\ \color{blue}{0} & \color{blue}{0} & 0 & \color{blue}{0} & \color{red}{4} \end{pmatrix} \qquad \begin{pmatrix} \color{red}{1} & 1 & \color{blue}{0} & 1 \\ \color{blue}{0} & 0 & \color{red}{1} & 1 \\ \color{blue}{0} & 0 & \color{blue}{0} & 0 \end{pmatrix}$$

Solution.

The first, second, and fourth matrices are in reduced row echelon form; the third matrix is in row echelon form. The pivots are in red; the other entries in the pivot columns are in blue.

3. Find the parametric form of the solutions of following system of equations in x_1, x_2, and x_3 by putting an augmented matrix into reduced row echelon form. State which variables (if any) are free variables. Describe the solution set geometrically.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ - 3x_2 - 6x_3 &= -3. \end{aligned}$$

Solution.

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{array}\right) \xrightarrow{R_2=R_2+4R_1} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & -3 & -6 & -3 \end{array}\right)$$

$$\xrightarrow{R_3=R_3+R_2} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1=R_1-R_2} \left(\begin{array}{ccc|c} 1 & 0 & -5 & -2 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_2=R_2\div 3} \left(\begin{array}{ccc|c} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The variables x_1 and x_2 correspond to pivot columns, but x_3 is free.

$$x_1 = -2 + 5x_3, \qquad x_2 = 1 - 2x_3, \qquad x_3 = x_3 \quad (x_3 \text{ real}).$$

This consistent system in three variables has one free variable, so the solution set is a line in $\mathbf{R}^3$.