1. Every color on my computer monitor is a vector in $\mathbb{R}^3$ with coordinates between 0 and 255, inclusive. The coordinates correspond to the amount of red, green, and blue in the color.

Given colors $v_1, v_2, \ldots, v_p$, we can form a “weighted average” of these colors by making a linear combination

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

with $c_1 + c_2 + \cdots + c_p = 1$. Example:

$$\frac{1}{2} \begin{pmatrix} 240 \\ 140 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ 120 \\ 100 \end{pmatrix} = \begin{pmatrix} 60 \\ 125 \\ 75 \end{pmatrix}$$

Consider the colors on the right. Are these colors linearly independent? What does this tell you about the colors?

After doing this problem, check out the interactive demo, where you can adjust sliders to find a prescribed color.

**Solution.**

The vectors

$$\begin{pmatrix} 240 \\ 140 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 120 \\ 100 \end{pmatrix}, \begin{pmatrix} 60 \\ 125 \\ 75 \end{pmatrix}$$

are linearly independent if and only if the vector equation

$$x \begin{pmatrix} 240 \\ 140 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 120 \\ 100 \end{pmatrix} + z \begin{pmatrix} 60 \\ 125 \\ 75 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is satisfied for some scalars $x, y, z$. This corresponds to a system of linear equations:

$$240x + 140y + 60z = 0$$
$$140x + 120y + 125z = 0$$
$$0x + 100y + 75z = 0$$
has only the trivial solution. This translates into the matrix (we don’t need to augment since it’s a homogeneous system)

\[
\begin{pmatrix}
240 & 0 & 60 \\
140 & 120 & 125 \\
0 & 100 & 75
\end{pmatrix}
\xrightarrow{\text{rref}}
\begin{pmatrix}
1 & 0 & .25 \\
0 & 1 & .75 \\
0 & 0 & 0
\end{pmatrix}
\xrightarrow{\text{parametric}}
\begin{align*}
x &= -.25z \\
y &= -.75z
\end{align*}

Hence the vectors are linearly dependent; taking \( z = 1 \) gives the linear dependence relation

\[
-\frac{1}{4} \begin{pmatrix} 240 \\ 140 \\ 0 \end{pmatrix} - \frac{3}{4} \begin{pmatrix} 0 \\ 120 \\ 100 \end{pmatrix} + \begin{pmatrix} 60 \\ 125 \\ 75 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

Rearranging gives

\[
\begin{pmatrix} 60 \\ 125 \\ 75 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 240 \\ 140 \\ 0 \end{pmatrix} + \frac{3}{4} \begin{pmatrix} 0 \\ 120 \\ 100 \end{pmatrix}.
\]

In terms of colors:

\[
\begin{array}{c}
\text{green} \\
\text{orange} \\
\text{blue}
\end{array}
= \frac{1}{4} \begin{array}{c}
\text{green} \\
\text{orange} \\
\text{blue}
\end{array} + \frac{3}{4} \begin{array}{c}
\text{green} \\
\text{orange} \\
\text{blue}
\end{array}.
\]

2. Circle TRUE if the statement is always true, and circle FALSE otherwise.

a) If \( A \) is a \( 3 \times 100 \) matrix of rank 2, then \( \dim(\text{Nul}(A)) = 97. \)

\[
\begin{array}{c}
\text{TRUE} \\
\text{FALSE}
\end{array}
\]

b) If \( A \) is an \( m \times n \) matrix and \( Ax = 0 \) has only the trivial solution, then the columns of \( A \) form a basis for \( \mathbb{R}^m \).

\[
\begin{array}{c}
\text{TRUE} \\
\text{FALSE}
\end{array}
\]

c) The set \( V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid x - 4z = 0 \right\} \) is a subspace of \( \mathbb{R}^4 \).

\[
\begin{array}{c}
\text{TRUE} \\
\text{FALSE}
\end{array}
\]

Solution.

a) False. By the Rank Theorem, \( \text{rank}(A) + \dim(\text{Nul}(A)) = 100 \), so \( \dim(\text{Nul}(A)) = 98. \)

b) False. For example, \( A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \) has only the trivial solution for \( Ax = 0 \), but its column space is a 2-dimensional subspace of \( \mathbb{R}^3 \).

c) True. \( V \) is \( \text{Nul}(A) \) for the \( 1 \times 4 \) matrix \( A \) below, and therefore is automatically a subspace of \( \mathbb{R}^4 \):

\[
A = \begin{pmatrix} 1 & 0 & -4 & 0 \end{pmatrix}.
\]

Alternatively, we could verify the subspace properties directly if we wished. This is much more work!
(1) The zero vector is in \( V \), since \( 0 - 4(0) = 0 \).

(2) Let \( u = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} \) and \( v = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix} \) be in \( V \), so \( x_1 - 4z_1 = 0 \) and \( x_2 - 4z_2 = 0 \).
We compute
\[ u + v = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \\ w_1 + w_2 \end{pmatrix}. \]
Is \( (x_1 + x_2) - 4(z_1 + z_2) = 0 \)? Yes, since
\[ (x_1 + x_2) - 4(z_1 + z_2) = (x_1 - 4z_1) + (x_2 - 4z_2) = 0 + 0 = 0. \]

(3) If \( u = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \) is in \( V \) then so is \( cu \) for any scalar \( c \):
\[ cu = \begin{pmatrix} cx \\ cy \\ cz \\ cw \end{pmatrix} \] and \( cx - 4cz = c(x - 4z) = c(0) = 0. \)

3. Let \( A = \begin{pmatrix} 1 & -5 & -2 & -4 \\ 2 & 3 & 9 & 5 \\ 1 & 1 & 4 & 2 \end{pmatrix} \), and let \( T \) be the matrix transformation associated to \( A \), so \( T(x) = Ax \).

a) What is the domain of \( T \)? What is the codomain of \( T \)? Give an example of a vector in the range of \( T \).

b) The RREF of \( A \) is \( \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \). Is there a vector in the codomain of \( T \) which is not in the range of \( T \)? Justify your answer.

**Solution.**

a) The domain is \( \mathbb{R}^4 \); the codomain is \( \mathbb{R}^3 \). The vector \( 0 = T(0) \) is contained in the range, as is
\[ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \]

b) Yes. The range of \( T \) is the column span of \( A \), and from the RREF of \( A \) we know \( A \) only has two pivots, so its column span is a 2-dimensional subspace of \( \mathbb{R}^3 \). Since \( \dim(\mathbb{R}^3) = 3 \), the range is not equal to \( \mathbb{R}^3 \).