Math 1553 Worksheet §4.4, Matrix Multiplication

Solutions

1. If \( A \) is a \( 3 \times 5 \) matrix and \( B \) is a \( 3 \times 2 \) matrix, which of the following are defined? Very briefly justify your answer.
   a) \( A - B \)
   b) \( AB \)
   c) \( A^T B \)
   d) \( B^T A \)
   e) \( A^2 \)

Solution.
Only (c) and (d).

a) \( A - B \) is nonsense. In order for \( A - B \) to be defined, \( A \) and \( B \) need to have the same number of rows and same number of columns as each other.

b) \( AB \) is undefined since the number of columns of \( A \) does not equal the number of rows of \( B \).

c) \( A^T \) is \( 5 \times 3 \) and \( B \) is \( 3 \times 2 \), so \( A^T B \) is a \( 5 \times 2 \) matrix.

d) \( B^T \) is \( 2 \times 3 \) and \( A \) is \( 3 \times 5 \), so \( B^T A \) is a \( 2 \times 5 \) matrix.

e) \( A^2 \) is nonsense (can’t do \( 3 \times 5 \) times \( 3 \times 5 \)).

2. True or false (justify your answer). Answer true if the statement is always true. Otherwise, answer false.
   a) Suppose \( A \) and \( B \) are matrices and the matrix product \( AB \) is defined. Then each column of \( AB \) must be a linear combination of the columns of \( A \).
   b) If \( A \) is a \( 3 \times 4 \) matrix and \( B \) is a \( 4 \times 2 \) matrix, then the linear transformation transformation \( Z \) defined by \( Z(x) = ABx \) has domain \( \mathbb{R}^2 \) and codomain \( \mathbb{R}^3 \).
   c) Suppose \( T : \mathbb{R}^3 \to \mathbb{R}^m \) and \( U : \mathbb{R}^m \to \mathbb{R}^p \) are linear transformations and \( U \circ T \) is onto. Then \( U \) and \( T \) must both be onto.

Solution.

a) True. If we let \( v_1, \ldots, v_p \) be the columns of \( B \), then \( AB = \left( Av_1 \ Av_2 \cdots Av_p \right) \), where \( Av_i \) is in the column span of \( A \) for every \( i \) (this is part of the definition of matrix multiplication of vectors).

b) True. In order for \( Bx \) to make sense, \( x \) must be in \( \mathbb{R}^2 \), and so \( Bx \) is in \( \mathbb{R}^4 \) and \( ABx \) is in \( \mathbb{R}^3 \). Therefore, the domain of \( Z \) is \( \mathbb{R}^2 \) and the codomain of \( Z \) is \( \mathbb{R}^3 \).

c) False. Take the linear transformations \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) and \( U : \mathbb{R}^3 \to \mathbb{R}^2 \) given by \( T(x, y, z) = (x, y, 0) \) and \( U(x, y, z) = (x, y) \). Then \( (U \circ T)(x, y, z) = (x, y) \), so
$U \circ T$ maps $\mathbb{R}^3$ onto $\mathbb{R}^2$. However, $T$ is not onto since the $z$-coordinate of every vector in its image is 0.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation clockwise by $60^\circ$. Let $U : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with standard matrix $\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$.

   a) Find the standard matrix for the composition $U \circ T$.

   b) Find the standard matrix for the composition $T \circ U$.

   c) Is rotating clockwise by $60^\circ$ and then performing $U$, the same as first performing $U$ and then rotating clockwise by $60^\circ$?

**Solution.**

a) The matrix for $T$ is $\begin{pmatrix} \cos(-60^\circ) & -\sin(-60^\circ) \\ \sin(-60^\circ) & \cos(-60^\circ) \end{pmatrix} = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$. The matrix for $U \circ T$ is $$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix} = \begin{pmatrix} -1 + \sqrt{3}/2 & 1/2 - \sqrt{3}/2 \\ 1/2 + \sqrt{3}/2 & -1 + \sqrt{3}/2 \end{pmatrix}.$$

b) The matrix for $T \circ U$ is $$\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 + \sqrt{3}/2 & 1/2 + \sqrt{3}/2 \\ 1/2 - \sqrt{3}/2 & -1 + \sqrt{3}/2 \end{pmatrix}.$$

c) No. In (a) and (b), we found that the standard matrices for $U \circ T$ and $T \circ U$ are different, so the transformations are different.