Supplemental problems: §3.2

1. Let $A$ be a $3 \times 4$ matrix with column vectors $v_1, v_2, v_3, v_4$, and suppose $v_2 = 2v_1 - 3v_4$. Consider the matrix transformation $T(x) = Ax$.

   a) Is it possible that $T$ is one-to-one? If yes, justify why. If no, find distinct vectors $v$ and $w$ so that $T(v) = T(w)$.

   b) Is it possible that $T$ is onto? Justify your answer.

Solution.

a) From the linear dependence condition we were given, we get

   $$-2v_1 + v_2 + 3v_4 = 0.$$

   The corresponding vector equation is just

   \[
   \begin{pmatrix}
   v_1 & v_2 & v_3 & v_4
   \end{pmatrix}
   \begin{pmatrix}
   -2 \\
   1 \\
   0 \\
   3
   \end{pmatrix}
   =
   \begin{pmatrix}
   0 \\
   0 \\
   0
   \end{pmatrix},
   \quad
   \text{so}
   \begin{pmatrix}
   -2 \\
   1 \\
   3
   \end{pmatrix}
   =
   \begin{pmatrix}
   0 \\
   0
   \end{pmatrix}.
   \]

   Therefore, $v = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix}$ and $w = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ both satisfy $Av = Aw = 0$, so $T$ cannot be one-to-one.

b) Yes. If $\{v_1, v_3, v_4\}$ is linearly independent then $A$ will have a pivot in every row and $T$ will be onto. Such a matrix $A$ is

   \[
   A = \begin{pmatrix}
   1 & 2 & 0 & 0 \\
   0 & 0 & 1 & 0 \\
   0 & -3 & 0 & 1
   \end{pmatrix}.
   \]

2. a) Which of the following are onto transformations? (Check all that apply.)

   \[\checkmark\] $T: \mathbb{R}^3 \to \mathbb{R}^3$, reflection over the $xy$-plane

   \[\square\] $T: \mathbb{R}^3 \to \mathbb{R}^3$, projection onto the $xy$-plane

   \[\checkmark\] $T: \mathbb{R}^3 \to \mathbb{R}^2$, project onto the $xy$-plane, forget the $z$-coordinate

   \[\checkmark\] $T: \mathbb{R}^2 \to \mathbb{R}^2$, scale the $x$-direction by 2

b) Let $A$ be a square matrix and let $T(x) = Ax$. Which of the following guarantee that $T$ is onto? (Check all that apply.)

   \[\checkmark\] $T$ is one-to-one
Ax = 0 is consistent

3. Find all real numbers h so that the transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by

$$T(v) = \begin{pmatrix} -1 & 0 & 2 - h \\ h & 0 & 3 \end{pmatrix} v$$

is onto.

**Solution.**

We row-reduce $A$ to find when it will have a pivot in every row:

$$\begin{pmatrix} -1 & 0 & 2 - h \\ h & 0 & 3 \end{pmatrix} \xrightarrow{R_2 = R_2 + hR_1} \begin{pmatrix} -1 & 0 & 2 - h \\ 0 & 0 & 3 + h(2 - h) \end{pmatrix}.$$ 

The matrix has a pivot in every row unless

$$3 + h(2 - h) = 0, \quad h^2 - 2h - 3 = 0, \quad (h - 3)(h + 1) = 0.$$ 

Therefore, $T$ is onto as long as $h \neq 3$ and $h \neq -1.$
Supplemental problems: §3.3

1. Circle T if the statement is always true, and circle F otherwise.

   a) T  F  If $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear and $T(e_1) = T(e_2)$, then the homogeneous equation $T(x) = 0$ has infinitely many solutions.

   b) T  F  If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a one-to-one linear transformation and $m \neq n$, then $T$ must not be onto.

Solution.

   a) True. The matrix transformation $T(x) = Ax$ is not one-to-one, so $Ax = 0$ has infinitely many solutions. For example, $e_1 - e_2$ is a non-trivial solution to $Ax = 0$ since $A(e_1 - e_2) = Ae_1 - Ae_2 = 0$.

   b) True. Let $A$ be the $m \times n$ standard matrix for $T$. If $T$ is both one-to-one and onto then $T$ must have a pivot in each column and in each row, which is only possible when $A$ is a square matrix ($m = n$).

2. Consider $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

   $$T(x, y, z) = (x, x + z, 3x - 4y + z, x).$$

   Is $T$ one-to-one? Justify your answer.

Solution.

One approach: We form the standard matrix $A$ for $T$:

$$A = \begin{pmatrix} T(e_1) & T(e_2) & T(e_3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & -4 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We row-reduce $A$ until we determine its pivot columns

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & -4 & 1 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 3 & -4 & 1 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - 3R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$A$ has a pivot in every column, so $T$ is one-to-one.

Alternative approach: $T$ is a linear transformation, so it is one-to-one if and only if the equation $T(x, y, z) = (0, 0, 0)$ has only the trivial solution.

If $T(x, y, z) = (x, x + z, 3x - 4y + z, x) = (0, 0, 0, 0)$ then $x = 0$, and

$$x + z = 0 \implies 0 + z = 0 \implies z = 0, \text{ and finally}$$

$$3x - 4y + z = 0 \implies 0 - 4y + 0 = 0 \implies y = 0,$$
so the trivial solution $x = y = z = 0$ is the only solution the homogeneous equation. Therefore, $T$ is one-to-one.

3. Which of the following transformations $T$ are onto? Which are one-to-one? If the transformation is not onto, find a vector not in the range. If the matrix is not one-to-one, find two vectors with the same image.

   a) The transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (y, y)$.

   b) JUST FOR FUN: Consider $T : \text{(Smooth functions)} \to \text{(Smooth functions)}$ given by $T(f) = f'$ (the derivative of $f$). Then $T$ is not a transformation from any $\mathbb{R}^n$ to $\mathbb{R}^m$, but it is still linear in the sense that for all smooth $f$ and $g$ and all scalars $c$ (by properties of differentiation we learned in Calculus 1):

   $$T(f + g) = T(f) + T(g) \quad \text{since} \quad (f + g)' = f' + g'$$

   $$T(cf) = cT(f) \quad \text{since} \quad (cf)' = cf'.$$

   Is $T$ one-to-one?

Solution.

   a) This is not onto. Everything in the range of $T$ has its first coordinate equal to its second, so there is no $(x, y, z)$ such that $T(x, y, z) = (1, 0)$. It is not one-to-one: for instance, $T(0, 0, 0) = (0, 0) = T(0, 0, 1)$.

   b) $T$ is not one-to-one. If $T$ were one-to-one, then for any smooth function $b$, the equation $T(f) = b$ would have at most one solution. However, note that if $f$ and $g$ are the functions $f(t) = t$ and $g(t) = t - 1$, then $f$ and $g$ are different functions but their derivatives are the same, so $T(f) = T(g)$. Therefore, $T$ is not one-to-one. It is not within the scope of Math 1553. If you find it confusing, feel free to ignore it.

4. In each case, determine whether $T$ is linear. Briefly justify.

   a) $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2, 1)$.

   b) $T(x, y) = (y, x^{1/3})$.

   c) $T(x, y, z) = 2x - 5z$.

Solution.

   a) Not linear. $T(0, 0) = (0, 0, 1) \neq (0, 0, 0)$.

   b) Not linear. The $x^{1/3}$ term gives it away. $T(0, 2) = (0, 2^{1/3})$ but $2T(0, 1) = (0, 2)$.

   c) Linear. In fact, $T(v) = Av$ where

   $$A = \begin{pmatrix} 2 & 0 & -5 \end{pmatrix}.$$
5. The second little pig has decided to build his house out of sticks. His house is shaped like a pyramid with a triangular base that has vertices at the points \((0, 0, 0), (2, 0, 0), (0, 2, 0),\) and \((1, 1, 1).\)

The big bad wolf finds the pig’s house and blows it down so that the house is rotated by an angle of \(45^\circ\) in a counterclockwise direction about the \(z\)-axis (look downward onto the \(xy\)-plane the way we usually picture the plane as \(\mathbb{R}^2\)), and then projected onto the \(xy\)-plane.

In the worksheet, we found the matrix for the transformation \(T\) caused by the wolf. Geometrically describe the image of the house under \(T\).

**Solution.**

Work shows that \(T(x) = Ax\), where

\[
A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

We know the house has been effectively destroyed, but what do its remains look like? To get an idea, let’s look at what happens to the vertices.

\[
\begin{align*}
\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, & \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix} \\
\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} &= \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix}, & \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \end{bmatrix}.
\end{align*}
\]

This indicates the pyramid has been squashed into a triangle in the \(xy\)-plane with vertices \(\begin{bmatrix} 0 \\ \sqrt{2} \\ -\sqrt{2} \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix}\). (the point \(\begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix}\) is along the top side of this triangle).

Effectively, the pyramid was rotated and then destroyed, so that its (rotated) base is all that remains.