Supplemental problems: §3.4

1. Consider \( T: \mathbb{R}^2 \to \mathbb{R}^3 \) defined by
   \[
   T \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x + 2y \\ 2x + y \\ x - y \end{array} \right)
   \]
and \( U: \mathbb{R}^3 \to \mathbb{R}^2 \) defined by first projecting onto the \( xy \)-plane (forgetting the \( z \)-coordinate), then rotating counterclockwise by \( 90^\circ \).
   a) Compute the standard matrices \( A \) and \( B \) for \( T \) and \( U \), respectively.
   b) Compute the standard matrices for \( T \circ U \) and \( U \circ T \).
   c) Circle all that apply:
      \( T \circ U \) is: one-to-one onto
      \( U \circ T \) is: one-to-one onto

2. Let \( T: \mathbb{R}^3 \to \mathbb{R}^2 \) be the linear transformation which projects onto the \( yz \)-plane and then forgets the \( x \)-coordinate, and let \( U: \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation of rotation counterclockwise by \( 60^\circ \). Their standard matrices are
   \[
   A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},
   \]
   respectively.
   a) Which composition makes sense? (Circle one.)
      \( U \circ T \quad T \circ U \)
   b) Find the standard matrix for the transformation that you circled in (b).

3. Find all matrices \( B \) that satisfy
   \[
   \begin{pmatrix} 1 & -3 \\ -3 & 5 \end{pmatrix} B = \begin{pmatrix} -3 & -11 \\ 1 & 17 \end{pmatrix}.
   \]

4. Let \( T \) and \( U \) be the (linear) transformations below:
   \( T(x_1, x_2, x_3) = (x_3 - x_1, x_2 + 4x_3, x_1, 2x_2 + x_3) \)
   \( U(x_1, x_2, x_3, x_4) = (x_1 - 2x_2, x_1) \).
   a) Which compositions makes sense (circle all that apply)? \( U \circ T \quad T \circ U \)
   b) Compute the standard matrix for \( T \) and for \( U \).
   c) Compute the standard matrix for each composition that you circled in (a).

5. True or false (justify your answer). Answer true if the statement is always true. Otherwise, answer false.
a) If $A$ and $B$ are matrices and the products $AB$ and $BA$ are both defined, then $A$ and $B$ must be square matrices with the same number of rows and columns.

b) If $A$, $B$, and $C$ are nonzero $2 \times 2$ matrices satisfying $BA = CA$, then $B = C$.

c) Suppose $A$ is an $4 \times 3$ matrix whose associated transformation $T(x) = Ax$ is not one-to-one. Then there must be a $3 \times 3$ matrix $B$ which is not the zero matrix and satisfies $AB = 0$.

d) Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^m \to \mathbb{R}^p$ are one-to-one linear transformations. Then $U \circ T$ is one-to-one. (What if $U$ and $T$ are not necessarily linear?)

6. In each case, use geometric intuition to either give an example of a matrix with the desired properties or explain why no such matrix exists.

a) A $3 \times 3$ matrix $P$, which is not the identity matrix or the zero matrix, and satisfies $P^2 = P$.

b) A $2 \times 2$ matrix $A$ satisfying $A^2 = I$.

c) A $2 \times 2$ matrix $A$ satisfying $A^3 = -I$. 
Supplemental problems: §3.5-3.6

1. a) Fill in: A and B are invertible $n \times n$ matrices, then the inverse of $AB$ is ________.

b) If the columns of an $n \times n$ matrix $Z$ are linearly independent, is $Z$ necessarily invertible? Justify your answer.

c) If $A$ and $B$ are $n \times n$ matrices and $ABx = 0$ has a unique solution, does $Ax = 0$ necessarily have a unique solution? Justify your answer.

2. Suppose $A$ is an invertible matrix and

$$A^{-1}e_1 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \quad A^{-1}e_2 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \quad A^{-1}e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$ 

Find $A$. 
