Supplemental problems: §5.4

1. True or false. Answer true if the statement is always true. Otherwise, answer false.
   a) If $A$ is an invertible matrix and $A$ is diagonalizable, then $A^{-1}$ is diagonalizable.
   b) A diagonalizable $n \times n$ matrix admits $n$ linearly independent eigenvectors.
   c) If $A$ is diagonalizable, then $A$ has $n$ distinct eigenvalues.

Solution.

a) True. If $A = PDP^{-1}$ and $A$ is invertible then its eigenvalues are all nonzero, so
   the diagonal entries of $D$ are nonzero and thus $D$ is invertible (pivot in every
diagonal position). Thus, $A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}$.
   
b) True. By the Diagonalization Theorem, an $n \times n$ matrix is diagonalizable if and
   only if it admits $n$ linearly independent eigenvectors.

   c) False. For instance, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is diagonal but has only one eigenvalue.

2. Give examples of $2 \times 2$ matrices with the following properties. Justify your answers.
   a) A matrix $A$ which is invertible and diagonalizable.
   b) A matrix $B$ which is invertible but not diagonalizable.
   c) A matrix $C$ which is not invertible but is diagonalizable.
   d) A matrix $D$ which is neither invertible nor diagonalizable.

Solution.

a) We can take any diagonal matrix with nonzero diagonal entries:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

b) A shear has only one eigenvalue $\lambda = 1$. The associated eigenspace is the $x$-
axis, so there do not exist two linearly independent eigenvectors. Hence it is
not diagonalizable.

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

c) We can take any diagonal matrix with some zero diagonal entries:

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

d) Such a matrix can only have the eigenvalue zero — otherwise it would have
two eigenvalues, hence be diagonalizable. Thus the characteristic polynomial
is \( f(\lambda) = \lambda^2 \). Here is a matrix with trace and determinant zero, whose zero-eigenspace (i.e., null space) is not all of \( \mathbb{R}^2 \):

\[
D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

3.  \( A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix} \).

a) Find the eigenvalues of \( A \), and find a basis for each eigenspace.

b) Is \( A \) diagonalizable? If your answer is yes, find a diagonal matrix \( D \) and an invertible matrix \( C \) so that \( A = CDC^{-1} \). If your answer is no, justify why \( A \) is not diagonalizable.

Solution.

a) We solve \( 0 = \det(A - \lambda I) \).

\[
0 = \det \begin{pmatrix} 2-\lambda & 3 & 1 \\ 3 & 2-\lambda & 4 \\ 0 & 0 & -1-\lambda \end{pmatrix} = (-1-\lambda)(-1)^6 \det \begin{pmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{pmatrix} = (-1-\lambda)((2-\lambda)^2 - 9) = (-1-\lambda)(\lambda^2 - 4\lambda - 5) = -(\lambda + 1)^2(\lambda - 5).
\]

So \( \lambda = -1 \) and \( \lambda = 5 \) are the eigenvalues.

\( \lambda = -1 \):

\[
\begin{pmatrix} 3 & 3 & 1 \\ 3 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 3 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 3 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 3 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\]

with solution \( x_1 = -x_2, x_2 = x_2, x_3 = 0 \). The \((-1)\)-eigenspace has basis \( \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\} \).

\( \lambda = 5 \):

\[
\begin{pmatrix} -3 & 3 & 1 \\ 3 & -3 & 4 \\ 0 & 0 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} -3 & 3 & 1 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} -3 & 3 & 1 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
\]

with solution \( x_1 = x_2, x_2 = x_2, x_3 = 0 \). The \(5\)-eigenspace has basis \( \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \).

b) \( A \) is a \( 3 \times 3 \) matrix that only admits 2 linearly independent eigenvectors, so \( A \) is not diagonalizable.
4. Let \( A = \begin{pmatrix} 8 & 36 & 62 \\ -6 & -34 & -62 \\ 3 & 18 & 33 \end{pmatrix} \).

The characteristic polynomial for \( A \) is \(-\lambda^3 + 7\lambda^2 - 16\lambda + 12\), and \( \lambda - 3 \) is a factor. Decide if \( A \) is diagonalizable. If it is, find an invertible matrix \( C \) and a diagonal matrix \( D \) such that \( A = CDC^{-1} \).

**Solution.**

By polynomial division,

\[
\frac{-\lambda^3 + 7\lambda^2 - 16\lambda + 12}{\lambda - 3} = -\lambda^2 + 4\lambda - 4 = -(\lambda - 2)^2.
\]

Thus, the characteristic poly factors as \(-(\lambda - 3)(\lambda - 2)^2\), so the eigenvalues are \( \lambda_1 = 3 \) and \( \lambda_2 = 2 \).

For \( \lambda_1 = 3 \), we row-reduce \( A - 3I \):

\[
\begin{pmatrix} 5 & 36 & 62 \\ -6 & -37 & -62 \\ 3 & 18 & 30 \end{pmatrix} \overset{R_1 \leftrightarrow R_3}{\rightarrow} \begin{pmatrix} 1 & 6 & 10 \\ -6 & -37 & -62 \\ 5 & 36 & 62 \end{pmatrix} \overset{R_2 = R_2 + 6R_1}{\rightarrow} \begin{pmatrix} 1 & 6 & 10 \\ 0 & -1 & -2 \\ 0 & 6 & 12 \end{pmatrix} \overset{R_3 = R_3 + 6R_2}{\rightarrow} \begin{pmatrix} 1 & 6 & 10 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Therefore, the solutions to \( (A - 3I \mid 0) \) are \( x_1 = 2x_3 \), \( x_2 = -2x_3 \), \( x_3 = x_3 \).

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}. \quad \text{The 3-eigenspace has basis } \left\{ \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}.
\]

For \( \lambda_2 = 2 \), we row-reduce \( A - 2I \):

\[
\begin{pmatrix} 6 & 36 & 62 \\ -6 & -36 & -62 \\ 3 & 18 & 31 \end{pmatrix} \overset{\text{rref}}{\rightarrow} \begin{pmatrix} 1 & 6 & \frac{31}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

The solutions to \( (A - 2I \mid 0) \) are \( x_1 = -6x_2 - \frac{31}{3}x_3 \), \( x_2 = x_2 \), \( x_3 = x_3 \).

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -6x_2 - \frac{31}{3}x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -\frac{31}{3} \\ 0 \\ 1 \end{pmatrix}. \quad \text{The 2-eigenspace has basis } \left\{ \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{31}{3} \\ 0 \\ 1 \end{pmatrix} \right\}.
Therefore, \( A = CDC^{-1} \) where
\[
C = \begin{pmatrix}
2 & -6 & -\frac{31}{3} \\
-2 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix} \quad D = \begin{pmatrix}
3 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix}.
\]

Note that we arranged the eigenvectors in \( C \) in order of the eigenvalues 3, 2, 2, so we had to put the diagonals of \( D \) in the same order.

5. Which of the following \( 3 \times 3 \) matrices are necessarily diagonalizable over the real numbers? (Circle all that apply.)

1. A matrix with three distinct real eigenvalues.
2. A matrix with one real eigenvalue.
3. A matrix with a real eigenvalue \( \lambda \) of algebraic multiplicity 2, such that the \( \lambda \)-eigenspace has dimension 2.
4. A matrix with a real eigenvalue \( \lambda \) such that the \( \lambda \)-eigenspace has dimension 2.

Solution.
The matrices in 1 and 3 are diagonalizable. A matrix with three distinct real eigenvalues automatically admits three linearly independent eigenvectors. If a matrix \( A \) has a real eigenvalue \( \lambda_1 \) of algebraic multiplicity 2, then it has another real eigenvalue \( \lambda_2 \) of algebraic multiplicity 1. The two eigenspaces provide three linearly independent eigenvectors.

The matrices in 2 and 4 need not be diagonalizable.

6. Suppose a \( 2 \times 2 \) matrix \( A \) has eigenvalue \( \lambda_1 = -2 \) with eigenvector \( \nu_1 = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} \), and eigenvalue \( \lambda_2 = -1 \) with eigenvector \( \nu_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \).

a) Find \( A \).

b) Find \( A^{100} \).

Solution.

a) We have \( A = CDC^{-1} \) where
\[
C = \begin{pmatrix}
3/2 & 1 \\
1 & -1
\end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix}
-2 & 0 \\
0 & -1
\end{pmatrix}.
\]

We compute \( C^{-1} = \frac{1}{-5/2} \begin{pmatrix}
-1 & -1 \\
-1 & 3/2
\end{pmatrix} = \frac{1}{5} \begin{pmatrix}
2 & 2 \\
2 & -3
\end{pmatrix}. \)

\[
A = CDC^{-1} = \frac{1}{5} \begin{pmatrix}
3/2 & 1 \\
1 & -1
\end{pmatrix} \begin{pmatrix}
-2 & 0 \\
0 & -1
\end{pmatrix} \begin{pmatrix}
2 & 2 \\
2 & -3
\end{pmatrix} = \frac{1}{5} \begin{pmatrix}
-8 & -3 \\
-2 & -7
\end{pmatrix}.
\]
b) 

\[ A^{100} = CD^{100}C^{-1} = \frac{1}{5} \begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix} \cdot D^{100} \begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} \]

\[ = \frac{1}{5} \begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2^{100} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} \]

\[ = \frac{1}{5} \begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \cdot 2^{100} & 2 \cdot 2^{100} \\ 2 & -3 \end{pmatrix} \]

\[ = \frac{1}{5} \begin{pmatrix} 3 \cdot 2^{100} + 2 & 3 \cdot 2^{100} - 3 \\ 2^{101} - 2 & 2^{101} + 3 \end{pmatrix}. \]

7. Suppose that \( A = C \begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix} C^{-1} \), where \( C \) has columns \( v_1 \) and \( v_2 \). Given \( x \) and \( y \) in the picture below, draw the vectors \( Ax \) and \( Ay \).

![Diagram](image)

**Solution.**

\( A \) does the same thing as \( D = \begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix} \), but in the \( v_1, v_2 \)-coordinate system. Since \( D \) scales the first coordinate by \( 1/2 \) and the second coordinate by \( -1 \), hence \( A \) scales the \( v_1 \)-coordinate by \( 1/2 \) and the \( v_2 \)-coordinate by \( -1 \).
Supplemental problems: §5.5

1. a) If $A$ is the matrix that implements rotation by $143^\circ$ in $\mathbb{R}^2$, then $A$ has no real eigenvalues.

b) A $3 \times 3$ matrix can have eigenvalues 3, 5, and $2+i$.

c) If $v = \begin{pmatrix} 2 + i \\ 1 \end{pmatrix}$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda = 1-i$, then $w = \begin{pmatrix} 2i - 1 \\ i \end{pmatrix}$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda = 1-i$.

Solution.

a) True. If $A$ had a real eigenvalue $\lambda$, then we would have $Ax = \lambda x$ for some nonzero vector $x$ in $\mathbb{R}^2$. This means that $x$ would lie on the same line through the origin as the rotation of $x$ by $143^\circ$, which is impossible.

b) False. If $2 + i$ is an eigenvalue then so is its conjugate $2 - i$.

c) True. Any nonzero complex multiple of $v$ is also an eigenvector for eigenvalue $1-i$, and $w = iv$.

2. Consider the matrix

$$A = \begin{bmatrix} 3\sqrt{3} - 1 & -5\sqrt{3} \\ 2\sqrt{3} & -3\sqrt{3} - 1 \end{bmatrix}$$

a) Find both complex eigenvalues of $A$.

b) Find an eigenvector corresponding to each eigenvalue.

Solution.

a) We compute the characteristic polynomial:

$$f(\lambda) = \det \begin{pmatrix} 3\sqrt{3} - 1 - \lambda & -5\sqrt{3} \\ 2\sqrt{3} & -3\sqrt{3} - 1 - \lambda \end{pmatrix}$$

$$= (-1 - \lambda + 3\sqrt{3})(-1 - \lambda - 3\sqrt{3}) + (2)(5)(3)$$

$$= (\lambda^2 - 9) + 10$$

$$= \lambda^2 + 2\lambda + 4.$$ 

By the quadratic formula,

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(4)}}{2} = \frac{-2 \pm \sqrt{4i}}{2} = -1 \pm \sqrt{3}i.$$ 

b) Let $\lambda = -1 - \sqrt{3}i$. Then

$$A - \lambda I = \begin{pmatrix} (i + 3)\sqrt{3} & -5\sqrt{3} \\ 2\sqrt{3} & (i - 3)\sqrt{3} \end{pmatrix}.$$
Since \( \det(A - \lambda I) = 0 \), the second row is a multiple of the first, so a row echelon form of \( A \) is
\[
\begin{pmatrix}
i + 3 & -5 \\
0 & 0
\end{pmatrix}.
\]

Hence an eigenvector with eigenvalue \(-1 - \sqrt{3}i\) is \( v = \begin{pmatrix} 5 \\ 3 + i \end{pmatrix} \). It follows that an eigenvector with eigenvalue \(-1 + \sqrt{3}i\) is \( \overline{v} = \begin{pmatrix} 5 \\ 3 - i \end{pmatrix} \).

3. Let
\[
A = \begin{pmatrix}4 & -3 & 3 \\ 3 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix}.
\]
Find all eigenvalues of \( A \). For each eigenvalue of \( A \), find a corresponding eigenvector.

**Solution.**
First we compute the characteristic polynomial by expanding cofactors along the third row:
\[
f(\lambda) = \det \left( \begin{pmatrix} 4 - \lambda & -3 & 3 \\ 3 & 4 - \lambda & -2 \\ 0 & 0 & 2 - \lambda \end{pmatrix} \right) = (2 - \lambda) \det \left( \begin{pmatrix} 4 - \lambda & -3 \\ 3 & 4 - \lambda \end{pmatrix} \right)
\]
\[
= (2 - \lambda)((4 - \lambda)^2 + 9) = (2 - \lambda)(\lambda^2 - 8\lambda + 25).
\]
Using the quadratic equation on the second factor, we find the eigenvalues
\[
\lambda_1 = 2 \quad \lambda_2 = 4 - 3i \quad \bar{\lambda}_2 = 4 + 3i.
\]
Next compute an eigenvector with eigenvalue \( \lambda_1 = 2 \):
\[
A - 2I = \begin{pmatrix} 2 & -3 & 3 \\ 3 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \overset{RREF}{\rightarrow} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.
\]
The parametric form is \( x = 0, \ y = z, \) so the parametric vector form of the solution is
\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = z \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \overset{eigenvector \ v_1}{\rightarrow} v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.
\]
Now we compute an eigenvector with eigenvalue \( \bar{\lambda}_2 = 4 - 3i \):
\[
A = (4 - 3i)I = \begin{pmatrix} 3i & -3 & 3 \\ 3 & 3i & -2 \\ 0 & 0 & 3i - 2 \end{pmatrix} \overset{R_1 \rightarrow R_1}{\rightarrow} \begin{pmatrix} 3i & -3 & 3 \\ 3 & 3i & -2 \\ 0 & 0 & 3i - 2 \end{pmatrix}
\]
\[
\overset{R_2 = R_2 - R_1}{\rightarrow} \begin{pmatrix} 3i & 0 & 0 \\ 3 & 3i & 0 \\ 0 & 0 & 3i - 2 \end{pmatrix} \overset{R_2 = R_2 + (3 + 2i)R_1}{\rightarrow} \begin{pmatrix} 3i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3i - 2 \end{pmatrix}
\]
\[
\overset{R_1 = R_1 + 3}{\rightarrow} \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
\]
The parametric form of the solution is $x = -iy, z = 0$, so the parametric vector form is

$$
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} =
\begin{pmatrix}
  -i \\
  1 \\
  0
\end{pmatrix}
$$

An eigenvector for the complex conjugate eigenvalue $\lambda_2 = 4 + 3i$ is the complex conjugate eigenvector $\bar{v}_2 = \begin{pmatrix}
  i \\
  1 \\
  0
\end{pmatrix}$. 