Supplemental problems: §5.6

1. Suppose the internet has four pages in the following manner. Arrows represent links from one page towards another. For example, page 1 links to page 4 but not vice versa.

![Diagram of internet links]

a) Write the importance matrix for this internet.

b) Assume there is no damping factor, so the importance matrix is the Google matrix. The 1-eigenspace is spanned by \[
\begin{pmatrix}
3/4 \\
3/4 \\
3/4 \\
1
\end{pmatrix}
\]. Find the steady-state vector for the Google matrix. What page has the highest rank?

Solution.

(a) The importance matrix is

\[
A = \begin{pmatrix}
0 & 0 & 1 & 0 \\
1/3 & 0 & 0 & 1/2 \\
1/3 & 0 & 0 & 1/2 \\
1/3 & 1 & 0 & 0
\end{pmatrix}
\]

(b) The steady-state vector is

\[
\begin{pmatrix}
1/3 \\
3/4 \\
3/4 \\
1
\end{pmatrix}
\] \[
\begin{pmatrix}
3/4 \\
3/4 \\
3/4 \\
1
\end{pmatrix}
\] = \[
\begin{pmatrix}
3/13 \\
3/13 \\
3/13 \\
4/13
\end{pmatrix}
\].

From the steady-state vector, we see page 4 has the highest rank.

2. The companies X, Y, and Z fight for customers. This year, company X has 40 customers, Company Y has 15 customers, and Z has 20 customers. Each year, the following changes occur:

- X keeps 75% of its customers, while losing 15% to Y and 10% to Z.
- Y keeps 60% of its customers, while losing 5% to X and 35% to Z.
- Z keeps 65% of its customers, while losing 15% to X and 20% to Y.
Write a stochastic matrix $A$ and a vector $x$ so that $Ax$ will give the number of customers for firms X, Y, and Z (respectively) after one year. You do not need to compute $Ax$.

**Solution.**

$$A = \begin{pmatrix} 0.75 & 0.05 & 0.15 \\ 0.15 & 0.6 & 0.20 \\ 0.1 & 0.35 & 0.65 \end{pmatrix}, \quad x = \begin{pmatrix} 40 \\ 15 \\ 20 \end{pmatrix}.$$
Supplemental problems: Chapter 6

1. True or false. If the statement is always true, answer true. Otherwise, answer false. Justify your answer.

   a) Suppose \( W = \text{Span}\{w\} \) for some vector \( w \neq 0 \), and suppose \( v \) is a vector orthogonal to \( w \). Then the orthogonal projection of \( v \) onto \( W \) is the zero vector.

   b) Suppose \( W \) is a subspace of \( \mathbb{R}^n \) and \( x \) is a vector in \( \mathbb{R}^n \). If \( x \) is not in \( W \), then \( x - x_W \) is not zero.

   c) Suppose \( W \) is a subspace of \( \mathbb{R}^n \) and \( x \) is in both \( W \) and \( W^\perp \). Then \( x = 0 \).

   d) Suppose \( \hat{x} \) is a least squares solution to \( Ax = b \). Then \( \hat{x} \) is the closest vector to \( b \) in the column space of \( A \).

Solution.

   a) True. Since \( v \in W^\perp \), its projection onto \( W \) is zero.

   b) True. If \( x \) is not in \( W \) then \( x \neq x_W \), so \( x - x_W \) is not zero.

   c) True. Since \( x \) is in \( W \) and \( W^\perp \) it is orthogonal to itself, so \( ||x||^2 = x \cdot x = 0 \). The length of \( x \) is zero, which means every entry of \( x \) is zero, hence \( x = 0 \).

   d) False: \( A\hat{x} \) is the closest vector to \( b \) in \( \text{Col} \ A \).

2. Let \( W = \text{Span}\{v_1, v_2\} \), where \( v_1 = \begin{pmatrix}-1 \\ 2 \\ 1\end{pmatrix} \) and \( v_2 = \begin{pmatrix}1 \\ 2 \\ 3\end{pmatrix} \).

   a) Find the closest point \( w \) in \( W \) to \( x = \begin{pmatrix}0 \\ 14 \\ -4\end{pmatrix} \).

   Let \( A = \begin{pmatrix}-1 & 1 \\ 2 & 2 \\ 1 & 3\end{pmatrix} \). We solve \( A^T Av = A^T x \).

   \[
   A^T A = \begin{pmatrix}6 & 6 \\ 6 & 14\end{pmatrix} \quad A^T \begin{pmatrix}0 \\ 14 \\ -4\end{pmatrix} = \begin{pmatrix}24 \\ 16\end{pmatrix}.
   \]

   We find \( \begin{pmatrix}6 & 6 & 24 \\ 6 & 14 & 16\end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix}1 & 0 & 5 \\ 0 & 1 & -1\end{pmatrix} \), so \( v = \begin{pmatrix}5 \\ -1\end{pmatrix} \) and therefore

   \[
   w = Av = \begin{pmatrix}-1 & 1 \\ 2 & 2 \\ 1 & 3\end{pmatrix} \begin{pmatrix}5 \\ -1\end{pmatrix} = \begin{pmatrix}-6 \\ 8 \\ 2\end{pmatrix}.
   \]
b) Find the distance from \( w \) to \( \begin{pmatrix} 0 \\ 14 \\ -4 \end{pmatrix} \).

\[
||x - w|| = \left\| \begin{pmatrix} 0 \\ 14 \\ -4 \end{pmatrix} - \begin{pmatrix} -6 \\ 8 \\ 2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 6 \\ 6 \\ -6 \end{pmatrix} \right\| = \sqrt{36 + 36 + 36} = \sqrt{108} = 6\sqrt{3}.
\]

c) Find the standard matrix for the orthogonal projection onto \( \text{Span}\{v_1\} \).

\[
B = \frac{1}{v_1 \cdot v_1} v_1 v_1^T = \frac{1}{(-1)^2 + 2^2 + 1^2} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} -1 & 2 & 1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & -2 & -1 \\ -2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}
\]

d) Find the standard matrix for the orthogonal projection onto \( W \).

Let \( A = \begin{pmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{pmatrix} \). Since the columns of \( A \) are linearly independent, our projection matrix is \( A(A^TA)^{-1}A^T \). We already computed \( A^TA \) in part (a), so our matrix is

\[
A(A^TA)^{-1}A^T = \begin{pmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 6 & 6 & 14 \\ 6 & 14 & 6 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \frac{1}{48} \begin{pmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 14 & -6 & -6 \\ -6 & 6 & 6 \end{pmatrix} \begin{pmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.
\]

3. Find the least-squares line \( y = Mx + B \) that approximates the data points \((-2, -11), \ (0, -2), \ (4, 2)\).

**Solution.**

If there were a line through the three data points, we would have:

\[
(x = -2) \quad B + M(-2) = -11
\]

\[
(x = 0) \quad B + M(0) = -2
\]

\[
(x = 4) \quad B + M(4) = 2.
\]

This is the matrix equation \( \begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} B \\ M \end{pmatrix} = \begin{pmatrix} -11 \\ -2 \\ 2 \end{pmatrix} \).
Thus, we are solving the least-squares problem to $Av = b$, where

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} -11 \\ -2 \\ 2 \end{pmatrix}.$$ 

We solve $A^T A \hat{x} = A^T b$, where $\hat{x} = \begin{pmatrix} B \\ M \end{pmatrix}$.

$$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 20 \end{pmatrix},$$

$$A^T b = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 4 \end{pmatrix} \begin{pmatrix} -11 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ 30 \end{pmatrix}.$$ 

So $\hat{x} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$. In other words, $y = -5 + 2x$, or $y = 2x - 5$. 